Complex Function Theory

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November 10, 2015

1 Exponential function

Definition 1.

$$e^z = e^z\left(\cos y + i\sin y\right) \qquad z = x + iy$$

1. $e^z \in A\left(\mathbb{C}\right)$ is an entire function. We will check that the Cauchy Riemman equations apply.

   $$u = \Re\left(e^z\right) = e^x\cos y \qquad v = \Im\left(e^z\right) = \sin y$$

   Thus, after a short check we can see that

   $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

   Indeed hold in this case.

2. Clearly, $|e^z| = e^x$ and $\arg\left(e^z\right) = y$
3. Since $e^{z+2\pi i} = e^z$ we have the identity $e^{2\pi i} = 1$.
4. $e^{z_1+z_2} = e^{z_1}\cdot e^{z_2}$
5. $\left(e^z\right)' = e^z$

Claim 1. If $f \in A\left(\mathbb{C}\right)$ and $f' = f$ then $f(z) = C\cdot e^z$.

Proof. We will look at $g(z) = f(z)e^{-z}$ using the basic rules of differentiation we arrive at

$$g' = f'e^{-z} - fe^{-z} \overset{\left(e^{-z}\right)'=-e^{-z}}{=} 0 \Rightarrow g \equiv C \Rightarrow f = C\cdot e^z$$

□

After we have these function we can define the hyperbolic functions and the inverse functions.

Definition 2.

$$\sin z = \frac{e^{-iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = \frac{e^z - e^{iz}}{2} = -i\sin\left(iz\right) \qquad \cosh z = \frac{e^z + e^{-z}}{2} = \cos\left(iz\right)$$

$$\tan z = \frac{\sin z}{\cos z} \qquad \cos z \neq 0$$

$$\sin z = 0 \Leftrightarrow e^{iz} = e^{-iz} \Leftrightarrow e^{2iz} = 1 \Leftrightarrow z = \pi k, k \in \mathbb{Z}$$

In a similar fashion,

$$\cos z \Leftrightarrow z = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$$

1. $e^{2iz} = -1$
2. $\cos z = \sin\left(\frac{\pi}{2} - z\right)$

1.1 Trigonometric identities

$$\sin\left(\frac{\pi}{2}-z\right)=\cos z \qquad \cos\left(\frac{\pi}{2}-z\right)=\sin z$$

$$\sin(z+w)=\sin z\cos w+\cos z\sin w$$

$$\cos(z+w)=\cos z\cos w-\sin z\sin w$$

$$\sin^2+\cos^2 z=1$$

$$\cosh^2 z-\sinh^2=1$$

$$\cos z=\cos x+\cos(iy)-\sin x\sin(iy)=\cos x\cosh y-i\sin x\sinh y$$

$$\sin z=\sin x\cos(iy)+\cos x\cos(iy)=\sin x\cosh y+i\cos x\sinh y$$

$$|cosz|^2=\overbrace{\cos^2 z}^{1-\sin^2 z}\cosh^2 y+\sin^2 x\sinh^2 y=\cosh^2 y-\sin^2 x\overbrace{\left(\cosh^2 y-\sinh^2 y\right)}^{1}=\cosh^2 y-\sin^2 x=\cos^2 x+\sinh^2 y$$

Check that $|\sin z|^2=\sin^2 x+\sinh^2 y=-\cos x+\cosh^2 y$.

Definition 3.

$$\overbrace{\text{Log}}^{\text{The set}}\ z=\log|z|+i\ \overbrace{\text{Arg}}^{\text{The set}}\ z$$

$$e^w=z\Leftrightarrow w\in\text{Log}(z)$$

$$e^w=e^{\Re(w)}\big(\cos(\Im w)+i\sin(\Im w)\big)=\overbrace{e^{\log|z|}}^{\rho}(\cos\theta+i\sin\theta)=\rho(\cos\theta+i\sin\theta)=z$$

As a result of these identites, we arrive at the result that $z=\rho(\cos\theta+i\sin\theta)=\rho e^{i\theta} \qquad -\pi\le\theta\le\pi$

2 Branches

Definition 4 (Branch of an argument)**.** Let $G\subset\mathbb{C}$ be a domain. $\forall z\in G:\alpha\in C(\mathbb{C}),\alpha(z)\in Arg(z)$

Example 1.

- $G=\mathbb{C}\backslash(-\infty,0]$ and $-\pi<\alpha(z)<\pi$
- $G=\mathbb{C}\backslash[0,+\infty)$ then $0<\alpha<2\pi$
- If G is a half plane then the branch of the argument exists in G.

Definition 5 (Branch of a logarithm)**.** Let $G\subset C$ be a domain. $l\in C(G)$. The branch of a logarithm is defined as [1]

$$\forall z\in G:(z)\in\text{Log}(z)\Leftrightarrow e^{l(z)}=z$$

Claim 2. If l is a logarithm branch in G then $l\in A(G)$ and $l'(z)=\frac{1}{z}$

Proof. [2]

$$\lim_{z_2\to z_1}\frac{l(z_2)-l(z_1)}{z_2-z_1}=\lim_{z_2\to z_1}\frac{1}{\frac{z_2-z_1}{l(z_2)-l(z_1}}=\frac{1}{\lim_{z_2\to z_1}\frac{z_2-z_1}{l(z_2)-l(z_1}}=\frac{1}{z'(l)}=\frac{1}{e^l}=\frac{1}{z}$$

□

Claim 3. Let $f:G\xrightarrow[\text{one-to-one}]{\text{onto}}G'$ and $f\in A(G)$ then $g=f^{-1}\in A(G')$ where $g'(z)=\frac{1}{f'\big(g(z)\big)}$

[1] The branches of log and arg exist and don't exist together.

[2] Using the definition of $z(l)=e^l$ then $z'(l)=e^l$ (the inverse function which exists since l is continuous on the branch)

Proof. In the next lesson we will prove this. □

Definition 6. $f \in A(G)$, $f \neq 0$ in G. Then h is a branch of $\log f$ if $h \in C(G)$ and $e^h = f(f(z) = z)$.

Claim 4. If a branch of $g = \log f$ exists in G then $g \in A(G)$ nad $g' = \frac{f'}{f}$.

Proof. $z_0 \in G$, $f(z_0) \neq 0$ Then by the continuity of f there exists a neighboorhood of u_{z_0} such that $f(u_{z_0}) \subset$ half plane then $g = \log \circ f \Rightarrow g \in A(u_{z_0})$ Leading us to $g' = \frac{f'}{f}$. □

Definition 7.

$$z^a := e^{a \log z} \qquad -\pi < \arg(z) < \pi \qquad a \in \mathbb{C}$$

This function is **analytic** in G

- $|z^a| = \left|e^{a \log z}\right| = e^{\Re(a \log z)} = e^{a\Re(\log z)} = e^{a \log|z|} = |z|^a$
- $\arg(z^a) = \Im(a \log z) = a\Im(\log z) = a \arg(z)$

Note 1. Notice that $-1 = i^2 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$. What did we do wrong? When working over the complex numbers, make sure that you define a valid branch.

We will now try to create a function such that

$$G = \hat{\mathbb{C}} \backslash [-1, 1] \xrightarrow[\text{1-1}]{\text{onto}} \Pi_+ \left\{w : \Re(w) > 0\right\}$$

We can create a function $f(z) = \frac{z+1}{z-1}$. It is easy to see that $1 \mapsto \infty$, $-1 \mapsto 0$ and forall $x \in (-1, 1) \Rightarrow \frac{x+1}{x-1} \in (-\infty, 0)$. Thus, the function we are looking for is

$$g(z) = \sqrt{\frac{z+1}{z-1}}$$

The root is because this allows our function to be **on** the whole line.

$$\frac{\sqrt{\frac{z+1}{z-1}} - 1}{\sqrt{\frac{z+1}{z-1}} + 1} = \frac{\sqrt{z+1} - \sqrt{z-1}}{\sqrt{z+1} + \sqrt{z-1}} = z - \sqrt{z^2 - 1}$$

Then we need to take the inverse function $z = \frac{1}{2}\left(1 + \frac{1}{w}\right)$.