Complex Function Theory

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In this lesson we will show that z^* is a reflection, e.g. for all $\{z_1, z_2, z_3\} \subset C$ where C is a line or a circle (we will seperate these two cases)

$$\left(z_1, z_2, z_3, z^*\right) = \overline{(z_1, z_2, z_3, z)}$$

Proof. 1. If C is a line, $z_1 = \infty$

$$\frac{z^* - z_2}{z_3 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_3 - \bar{z}_2}$$

The distances are equal.

$$|z_3 - z_2| = |\bar{z}_3 - \bar{z}_2| \Rightarrow \left|z^* - z_2\right| = |z - z_2|$$

All that is left to prove is that

$$\left[z, z^*\right] \perp [z_2, z_3] \qquad \text{and} z - z^* \perp z_2 - z_3$$

Using the properties for the inner product, for any $w_1 = (u_1, v_1) \qquad w_2 = (u_2, v_2)$ we have

$$\langle w_!, w_2\rangle = u_1u_2 + v_1v_2 = \Re\left(w_1\bar{w}_2\right)$$

Does $\Re\left[\left(z - z^*\right)\overline{(z_2 - z_3)}\right]$ equal to 0? Yes.

$$\left(z - z^*\right)\left(\bar{z}_2 - \bar{z}_3\right) = \overbrace{(z - z_2)\overline{(z_2 - z_3)}}^{=\overline{(z^* - z_2)}(z_2 - z_3)} + \left(z_2 - z^*\right)\overline{(z_2 - z_3)} = (z_2 - z_3)\overline{(z^* - z_2)} + \left(z_2 - z^*\right)\overline{(z_2 - z_3)} \in \mathbb{R}i$$

And therefore $\Re(\dots) = 0$, as needed.

Explanation: Let $w_1, w_2 \in \mathbb{C}$. Then $w_1\bar{w}_2 - \bar{w}_1 w_2 \in \mathbb{R}$. In addition, z_1, z_2 are in different halves of the plane. This is because the function $\varphi(z) = \frac{z - z_2}{z_3 - z_2}$ sends $(\infty, z_2, z_3) \mapsto (\infty, 0, 1)$ and for all values in $\mathbb{C}$ we have $\varphi : \mathbb{C} \hookrightarrow \mathbb{R}$, then we are done.

2. In the second case, if C is a circle with a at the center and a radius of r.
Claim 1. $(z^* - a)\overline{(z - a)} = r^2$.

Proof. Using the fact that Möbius transformation keep the cross-product:

$$\overline{(z_1, z_2, z_3, z)} = \overline{(z_1 - a, z_2 - a, z_3 - a, z - a)} = \left(\frac{r^2}{z_1 - a}, \frac{r^2}{z_2 - a}, \frac{r^2}{z_3 - a}, \overline{z - a}\right)$$
$$= \left(z_1 - a, z_2 - a, z_3 - a, \frac{r^2}{\overline{\overline{z - a}}}\right) = \left(z_1, z_2, z_3, a + \frac{r^2}{\bar{z} - \bar{a}}\right) \to z^* = a + \frac{r^2}{\bar{z} - \bar{a}}$$

Where the third equality comes from the fact that for all $z_j \in C$ we have $\left(z_j - a\right)\overline{\left(z_j - a\right)} = \left|z_j - a\right| = r^2 \Rightarrow \overline{z_j - a} = {r^2}/{z_j - a}$ Inferring from that we arrive at:

(a) $|z^*||z-a| = r^2 \Rightarrow |z^*-a| < r \Leftrightarrow |z-a| > r$

(b) $\frac{z^*-a}{z-a} = \frac{r^2}{|z-a|^2} > 0 \Rightarrow z_1, z^*$ are on the same ray from the center of the circle.

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Example 1. Using the group $M\ddot{o}b\,(\mathbb{C})$ there are a few interseting sub-groups.

1. $\mathbb{C}_+(=\mathbb{H}) = \{\Im(z) > 0\}$. We are looking at the Möbius transformations which are one-to-one and onto over this group. e.g. $\mathbb{C}_+ \xrightarrow[1-1]{\text{onto}} \mathbb{C}_+ \Rightarrow \mathbb{R} \to \mathbb{R}$ This sub-group can be showed as

$$h_A(z)\, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; a,b,c,d \in \mathbb{R} \qquad \det A = ad - bc = 1 \Rightarrow SL_2\,(\mathbb{R})$$

$$\Im\left(h_A(z)\right) = \Im\left(\frac{az+b}{cz+d}\right) = \frac{\Im\left[(az+b)\,(c\bar{z}+d)\right]}{|cz+d|^2} = \frac{\Im\left(ac|z|^2 + adz + bc\bar{z} + bd\right)}{|cz+d|^2}$$

$$= \frac{ad\Im(z) - bc\Im(z)}{|cz+d|^2} = \frac{\Im(z)}{|cz+d|^2} \Rightarrow \; h_A : \begin{array}{ccc} \mathbb{C}_+ & \xrightarrow{1-1} & \mathbb{C}_+ \\ \mathbb{R} & \longrightarrow & \mathbb{R} \\ \mathbb{C}_- & \xrightarrow{onto} & \mathbb{C}_- \end{array}$$

2. $\mathbb{D} = \{|z| < 1\} \qquad h_A : \mathbb{D} \xrightarrow[1-1]{onto} h_A(z) = \frac{az+b}{\bar{b}z+\bar{a}}$

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \qquad a,b \in \mathbb{C} \qquad |a| - |b| = 1$$

Check:

$$1 - \left|h_A(z)\right|^2 = \frac{\left|\bar{b}z - \bar{a}\right|^2 - |az+b|^2}{\left|\bar{b}z + \bar{a}\right|^2} \stackrel{?}{=} \frac{1-|z|^2}{\left|\bar{b}z + \bar{a}\right|^2}$$

Note 1.

$$\frac{az+b}{\bar{b}z+\bar{a}} = \frac{a}{\bar{a}}\frac{z + b/a}{1 + \overline{b/a}} = \lambda\frac{z - z_0}{1 - z\bar{z}_0}$$

Where $\lambda := \frac{a}{\bar{a}}$ $(|\lambda| = 1)$, $h_A\,(z_0) = 0$ and $z_0 : -b/a$. Here λ is parameter representing the rotation and z_0 is an axis around which the rotation occurs. ($|z_0| < 1$ since $|b| < |a|$). An interesting map is the Cayley transform from the positive half of the complex plane to the unit disk:

$$z \mapsto \frac{z-i}{z+i} \qquad \mathbb{C}_+ \hookrightarrow \mathbb{D}$$