Complex Function Theory

Mikhail Sodin
Arazim ©

November 3, 2015

1 Möbius transformation

$$h_A(z) = \frac{az+b}{cz+d} \qquad (c,d) \neq (0,0) \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$h'_A(z) = \frac{\det A}{(cz+d)^2} \qquad \det A \neq 0$$

$$c \neq 0 \quad h_A\left(-\tfrac{d}{c}\right) = \infty$$

$$h_A(\infty)\, \tfrac{a}{c}$$

$$c = 0 \quad h_A(z) = az + b$$

$$h_A(\infty)\, \infty$$

$$A = \lambda I, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (Id) \qquad h_A\{\lambda I\}(z) = z$$

1.1 Projective space

In the space $\mathbb{C}^2 \setminus \{0\}$ we can say that two vectors Z and W are equivalent $(Z \sim W)$ if there are two scalars, $a, b \in \mathbb{C}$ such that $\alpha Z + \beta W = 0 \Rightarrow W = \lambda Z, \lambda \in \mathbb{C}\setminus\{0\}$.
The space of equivalence relations over $\mathbb{C}$ is marked as $P(\mathbb{C})$
For example,

$$\begin{array}{ccc} \mathbb{C}^2 & \underset{A}{\rightarrow} & \mathbb{C}^2 \\ \downarrow \pi & & \downarrow \pi \\ P\left(\mathbb{C}^2\right) & \xrightarrow{h_A} & P\left(\mathbb{C}^2\right) \end{array}$$

$$\pi : [z_1, z_2] \mapsto \frac{z_1}{z_2}\,(z_2 \neq 0) \qquad \pi\,[z_1, 0] \mapsto \infty$$

Definition 1. $h_A := \pi \circ A \circ \pi^{-1}$ $\pi^{-1}(z) = [z, 1] = \begin{pmatrix} z \\ 1 \end{pmatrix}$

$$\left(A \circ \pi^{-1}\right)(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z & 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$$

$$\pi \circ \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \frac{az+b}{cz+d}$$

$$\Rightarrow h_A \circ h_B = \pi \circ A \circ \cancel{\pi^{-1} \circ \pi} \circ B \circ \pi^{-1} = \pi \circ A \circ B \circ \pi^{-1} = h_{AB}$$

1.2 Fixed points

- In the case of $c = 0$, then $h(z) = az + b$, $az + b = z$ the fixed points in this case are $\infty, \frac{-b}{a-1}$.
- if $c \neq 0$ then $\frac{az+b}{cz+d} = z \qquad cz^2 + (d-a)z - b = 0$
- Either way, $\# \{\text{fixed pts}\} \leq 2 \, (h \neq Id)$

1.3 Three-fold transitivity

Theorem 1. *Let (z_1, z_2, z_3) and (w_1, w_2, w_2) such that $w_i \neq w_j (i \neq j)$ and $z_i \neq z_j (i \neq j)$. THen there exists a unique $h \in Mob\,(\mathbb{C})$ such that $h : (z_1, z_2, z_3) \mapsto (w_1, w_2, w_3)$*

Proof. Uniqueness Let $h_1, h_2 : (z_1, z_2, z_3) \mapsto (w_1, w_2, w_2)$. We will define $h := h_2^{-1} \circ h_1 h(z_i) = z_i \Rightarrow h = id \Rightarrow h_1 = h_2$

Existence $\varphi\,(z_1, z_2, z_3) \mapsto (\infty, 0, 1)$

$$\varphi\,(z) = \frac{z - z_2}{z - z_1} : \frac{z_3 - z_2}{z_3 - z_1}$$

$$\varphi_1 = (z_1, z_2, z_3) \mapsto (w_1, w_2, w_3) \qquad \varphi_2 = (z_1, z_2, z_3) \mapsto (w_1, w_2, w_3)$$

And the required transformation is $h = \varphi_2^{-1} \circ \varphi_1$

□

1.4 Famous transformations

$$z \mapsto kz \qquad k > 0 \quad \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \quad \text{homothety}$$

$$z \mapsto \lambda z \qquad |\lambda| = 1 \quad \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \quad \text{rotation}$$

$$z \mapsto z + b \qquad b \in \mathbb{C} \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{translation}$$

$$z \mapsto {}^1\!/_z \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{inversion}$$

Theorem 2. *Every möbius transformation is a composition of elementary transformations.*

Proof.

$$h(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}\,(cz+d) + b - \frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{z+d} \qquad k\lambda = c \neq 0 \qquad k|c|\ \lambda = \frac{c}{|c|}$$

$$z_1 = kz \qquad z_2 = \lambda z_1 \qquad z_3 = z_2 + d \qquad z_4 = \frac{1}{3} \ldots z_7 = z_6 + \frac{a}{c}$$

□

Definition 2 (extended circle)**.** An extended circle is a circle or a line (passes throught the north pole).

Theorem 3. *Möbius transformations keep extended circles.*

Proof. It is enough to show that this is true for $h(z) = \frac{1}{z}$.

$$\alpha \cdot |z|^2 + \beta \cdot \Re(z) + \gamma \cdot \Im(z) + \delta = 0 \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

$$\alpha \cdot \left|\frac{1}{z}\right|^2 + \beta \cdot \Re\left(\frac{1}{z}\right) + \gamma \cdot \Im\left(\frac{1}{z}\right) + \delta = 0$$

$$\frac{\alpha}{|z|^2} + \beta \cdot \frac{\Re(z)}{|z|^2} - \gamma \cdot \frac{\Im(z)}{|z|^2} + \delta = 0$$

$$\delta \cdot |z|^2 + \beta \cdot \Re(z) - \gamma \cdot \Im(z) + \alpha = 0$$

□

2 Cross-ratio

$$\varphi(z) = \frac{z - z_2}{z - z_1} \cdot \frac{z_3 - z_2}{z_3 - z_1}$$

Definition 3. $(z_1, z_2, z_3, z_4) \mapsto \varphi(z_4) \frac{z_4 - z_2}{z_4 - z_1} : \frac{z_3 - z_2}{z_3 - z_1}$ Where $z_i \neq z_j$ if $i \neq j$. $(\infty, z_2, z_3, z_4) = \frac{z_2 - z_4}{z_2 - z_3}$

$$(z_1, z_2, z_3, z_1) = \infty$$

Theorem 4.

$$\forall h \in M\ddot{o}b(\mathbb{C}) \,.\, \big(h(z_1), h(z_2), h(z_3), h(z_4)\big) = (z_1, z_2, z_3, z_4)$$

Proof. $\varphi \circ h^{-1} : h(z_1) \mapsto \infty \qquad h(z_2) \mapsto 0 \qquad h_3 \mapsto 1$

$$\big(h(z_1), h(z_2), h(z_3), h(z_4) = \left(\varphi \circ h^{-1}\right)\big(h(z_4)\big) = \varphi(z_4)\,(z_1, z_2, z_3, z_4)$$

□

Corollary 1. *$w : z_i \to w_i$ then for $w = w(z)$ we have $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$.*

Theorem 5. *$\{z_1, z_2, z_3, z_4\}$ are on the same extended circle iff $(z_1, z_2, z_3, z_4) \in \mathbb{R}$.*

Proof. We will take an extended circle C such that $z_1, z_2, z_3 \in C$ (we know that this exists from high school). Also, by definition we know that $\varphi : C \to \mathbb{R}$. Since φ is one-to-one and onto, we have $(z_1, z_2, z_3, z_4) = \varphi(z_4) \in \mathbb{R} \Leftrightarrow \varphi(z_4) \in \varphi(C) \Leftrightarrow z_4 \in C$ □

3 Symmetry using extended circles

The ideas expressed in this section are mostly geometrical, and as such the write of these notes ahd trouble copying them down. For further refference, hopefully we will find handwritten notes.

Definition 4. $\varphi(C) = \mathbb{R}$, $z^* \in C \Leftrightarrow z = z^*$, $\varphi(z^*) = \overline{\varphi(z)}$.

We will assume that $\psi(C) = \mathbb{R}$, does $\psi(z^*) = \overline{\psi(z)}$?
We will define $w := \varphi(z)\ \bar{w} = \varphi(z^*)$ and $z = \varphi^{-1}(w)$ then $\psi(z) = \left(\psi \circ \varphi^{-1}\right)(w) \qquad \psi(z^*) = \left(\psi \circ \varphi^{-1}\right)(\bar{w})$

$$\psi \circ \varphi^{-1}(\mathbb{R}) = \mathbb{R} \Rightarrow \left(\psi \circ \varphi^{-1}\right)(w) = \frac{\alpha w + \beta}{\gamma w + \delta}\ \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

$$\left(\psi \circ \varphi^{-1}\right)(\bar{w}) = \frac{a\bar{w} + \beta}{\gamma \bar{w} + \delta} = \overline{\frac{\alpha w + \beta}{\gamma w + \delta}} = \overline{(\psi \circ \varphi)(w)}$$

Corollary 2. *$\{z_1, z_2, z_3\} \subset C$ then $z' = z^*$ iff $(z_1, z_2, z_3, z_4) = \overline{(z_1, z_2, z_3, z_4)}$*

Corollary 3. *Möbius transformations keep the symmetry. $C' = h(C), C$. z_1, z^* are symmetric over $C \Rightarrow h(z), h(z^*)$ are symmetric over $C' = h(C)$*