Complex Function Theory

Mikhail Sodin
Arazim ©

October 29, 2015

1 Continuation of arcs

- An arc $\gamma : I \to \mathbb{C}$ is said to be in C^1 if for all $t \in I$

$$\exists \lim_{t' \to t} \frac{\gamma\left(t'\right) - \gamma\left(t\right)}{t' - t} = \gamma'\left(t\right) = \dot{\gamma}\left(t\right)$$

- If there exists a countable set of points where the limit does not exist, the arc is said to be intermitenly continuous.
- $t_0 \in I$ is regular if $\dot{\gamma} \neq 0$.
- The angle between two arcs γ_1 and γ_2 is $\alpha = \arg \dot{\gamma}_2\left(t_2\right) - \arg \dot{\gamma}_1\left(t_1\right)$ $\left(\gamma_1\left(t_1\right) = \gamma_2\left(t_2\right)\right)$.

Theorem 1. *If $f'\left(z_0\right) \neq 0, f \in A\left(u_{z_0}\right)$ (u_{z_0} is a neighboorhood of z_0) then f keeps the angle between arcs intersecting at z_0*

Proof.
Claim 1.

$$\frac{d}{dt}\left(f \circ \gamma\right)\left(t\right) = f'\left(\gamma\left(t\right)\right)\dot{\gamma}\left(t\right)$$

$$\begin{aligned}\beta &= \arg\left[f'(z_0)\dot{\gamma}_2\left(t_2\right)\right] - \arg\left[f'(z_0)\dot{\gamma}_1\left(t_2\right)\right] = \cancel{\arg f'\left(z_0\right)} + \arg\dot{\gamma}_2\left(t_2\right) - \cancel{\arg f'\left(z_0\right)} - \arg\dot{\gamma}_1\left(t_1\right) \\ &= \arg\dot{\gamma}_2\left(t_2\right) - \arg\dot{\gamma}_1\left(t_1\right) = \alpha\end{aligned}$$

□

Definition 1. $f : \mathbb{R}^2 \supset G \to \mathbb{R}^2$, $z_0 = (x_0, y_0) \in G$
f is a **conformal map** in z_0 if f keeps the angle between two arcs at z_0.

Theorem 2. *If $f \in C^1\left(G\right)$ is a conformal map in all of G, then $f \in A\left(G\right)$ and $f'\left(z\right) \neq 0$ in G.*

Lemma 1. *A special case of this is in the linear transformation $Tz = az + b\bar{z}$ $(a, b \in \mathbb{C})$. If T is conformal then $b = 0$.*

of the special case.

- $a + b \neq 0$ (otherwise $T\mathbb{R} = \{0\}$).
- $a \neq 0$ (otherwise $Tz = b\bar{z}$ "anti-conformal").

□

Let λ be a point on the unit circle.

$$\arg\lambda = \arg\left(a\lambda + b\bar{\lambda}\right) - \arg\left(a+b\right)$$

$$0 = \arg\left(a + b\frac{\bar{\lambda}}{\lambda}\right) - \arg\left(a+b\right)$$

$$\forall\lambda.|\lambda| = 1.\arg\left(a + b\frac{\bar{\lambda}}{\lambda}\right) = \arg\left(a+b\right)$$

$$\left|b\frac{\bar{\lambda}}{\lambda}\right| = |b| \Rightarrow |b| = 0 \Rightarrow b = 0$$

General case. $\gamma\left(t_0\right) = z_0, \gamma\left(t\right) = x(t) + iy(t)$

$$\frac{d}{dt}\left(f\circ\gamma\right)(t) = \frac{\partial f(z_0)}{\partial x}\dot{x} + \frac{\partial f(z_0)}{\partial y} = \frac{1}{2}\frac{\partial f(z_0)}{\partial x}\left(\dot{\gamma}\left(t_0\right) + \bar{\dot{\gamma}}(t_0)\right) + \frac{1}{2i}\frac{\partial f(z_0)}{\partial y}\left(\dot{\gamma}\left(t_0\right) + \bar{\dot{\gamma}}(t_0)\right)$$

$$= \overbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x}\left(z_0\right) - i\frac{\partial f}{\partial y}(z_0)\right)}^{=:a}\cdot\dot{\gamma}\left(t_0\right) + \overbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x}\left(z_0\right) + i\frac{\partial f}{\partial y}(z_0)\right)}^{=:b=\frac{\partial f}{\partial z}}\dot{\bar{\gamma}}\left(t_0\right) = a\dot{\gamma}\left(t_0\right) + b\bar{\dot{}}(t_0)$$

f is conformal at $z_0 \Rightarrow TZ = aZ + b\bar{Z}$ is conformal in $Z = 0$. □

2 Distortion

$\gamma : [a,b] \to \mathbb{C}$

$$L\left(\gamma\right) = \int_b^a \underbrace{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}_{\left|\dot{\gamma}(t)\right|}dt = \int_a^b \left|\dot{\gamma}(t)\right| dt$$

$$L\left(f\circ\gamma\right) = \int_a^b \left|\frac{d}{dt}\left(f\circ\gamma\right)(t)\right| dt = \int_a^b \left|f'(\gamma(t))\right|\left|\dot{\gamma}(t)\right| dt$$