Complex Function Theory

Mikhail Sodin
Arazim ©

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1 Cauchy-Riemman

Let $G \subset \mathbb{C}$ be an open set and let $f : G \to \mathbb{C}$ be differentiable at $z \in G$ if

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

Claim 1. $f = u + iv$ is differentiable at z. Then there exists partial derivatives of u and v at $z = x + yi$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Proof. We will take a $h = k \in \mathbb{R}$.

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+k,y) - u(x,y)}{k} + i \cdot \frac{v(x+k,y) - v(x,y)}{k} \underset{k\to 0}{\overset{u}{\longrightarrow}}{}_x + iv_x$$

$$f'(z) = \frac{\partial f}{\partial x} = u_x + iv_x$$

$$f'(z) = \lim_{l\to 0} \frac{f(z+il) - f(z)}{il} = -i\frac{\partial f}{\partial y} = -i\left(u_y + iv_y\right) = v_y - iu_y = u_x + iv_x$$

$\Rightarrow$ The Cauchy Riemann equation hols. □

Definition 1. f is analytic[1] in G if it is differentiable ($\mathbb{C}$) in all of G.

Theorem 1. *1. If $f \in A(G)$, $f = u + iv$ then the C-R equations hold in G.*

2. $u, v \in C^1(G)$, then the C-R equations hold and $f = u + iv \in A(G)$.

Proof.

$$u(x+k, y+l) = u(x,y) + \overbrace{k \cdot u_x(x,y)}^{=\alpha} + \overbrace{l \cdot u_y(x,y)}^{=\beta} + \varepsilon_1$$

$$v(x+k, y+l) = v(x,y) + \overbrace{k \cdot v_x(x,y)}^{=-\beta} + \overbrace{l \cdot v_y(x,y)}^{=\alpha} + \varepsilon_2 \qquad \varepsilon_1, \varepsilon_2 = o(h)$$

$$\begin{aligned} f(z+h) =& u(x+k,y+l) + iu(x+k,y+l) = f(z) + k\alpha + l\beta + i(-k\beta + l\alpha) + \varepsilon_i + i\varepsilon_2 \\ &= f(z) + k(\alpha - i\beta) + i(\alpha - i\beta) + \varepsilon_1 + \varepsilon_2 = f(z) + (k + il)(\alpha + i\beta) + \varepsilon_1 + \varepsilon_2 \\ &= f(z) + h(\alpha - i\beta) + \varepsilon \end{aligned}$$

As a result,

$$\frac{f(z+h) - f(z)}{h} = (\alpha - i\beta) + \frac{\varepsilon}{h} \Rightarrow \frac{f\left(z + h - f(z)\right)}{h} \overset{h\to 0}{=} (\alpha - i\beta)$$

□

[1]Sometimes refered to as holomorphic and can be marked as $A(G)$ or $H(G)$

Theorem 2. *$G \subset \mathbb{C}$ is a domain. The following are equivalent:*

1. *$f \equiv const$.*
2. *$f' \equiv 0$.*
3. *$\Im(f) \equiv const$ (or $\Re(f) \equiv const$)*
4. *$|f| = const$*
5. *$arg(f) = const$.*

Proof. $(1) \Leftrightarrow (2)$ and $(1) \Rightarrow (2), (4), (5)$ are obvious.

$(3) \Rightarrow (1)$ $v = const \Rightarrow u_x, u_y \equiv 0 + \text{calc } 2 \Rightarrow u = const$.

$(4) \Rightarrow (1)$ $u^2 + v^2 = const$ In the first case, if $const = 0$ then $f \equiv 0$. On the other hand, if $const \neq 0$ then

$$\begin{cases} u \cdot u_x + v \cdot v_x = 0 \\ u \cdot u_y + v \cdot v_y = 0 \end{cases} \Rightarrow \begin{cases} u \cdot u_x + v \cdot v_x = 0 \\ -u \cdot v_x + v \cdot u_x = 0 \end{cases} \Rightarrow \begin{vmatrix} u & v \\ -v & u \end{vmatrix} = u^2 + v^2 \neq 0$$

Which means that in all of G, $u_x = v_x = 0 \Rightarrow u_y = v_y = 0 \Rightarrow f \equiv const$.

$(5) \Rightarrow (1)$ $v = ku$ in G, $k = const$ $0 = v - ku = \Re\left[-(k+i)(u+iv)\right] = -\Re\left[(k+i)f\right] \Rightarrow f \overset{(3)}{=} const$

□

Definition 2. $\partial_z := \frac{1}{2}\left(\partial_x - \partial_y\right)' = \frac{1}{2}\left(\partial_x + \frac{1}{i}\partial_y\right)$ $\partial_{\bar{z}} := \frac{1}{2}\left(\partial_x + i\partial_y\right)$

Note 1. $\partial_{\bar{z}} f = 0 \Leftrightarrow$ (C-R)

$$\partial_{\bar{z}} = \partial_{\bar{z}}(u + iv) = \frac{1}{2}\left(u_x + iu_y\right) + \frac{i}{2}\left(v_x + iv_y\right) = \frac{1}{2}\left(u_x - v_y\right) + \frac{i}{2}\left(u_y + v_x\right)$$

Definition 3 (Laplacian)**.**

$$\triangle = \partial_x^2 + \partial_y^2$$

Definition 4. $h : G \to \mathbb{C}$ is harmonic in G if $h \in C^2(G)$ and $\triangle h = 0$.

Note 2. $\triangle = 4\partial_{z\bar{z}}$

$$\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} = 4 \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \cdot \frac{1}{2}\left(\left(\frac{\partial}{\partial x}r + i\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}\right)\right.$$

Theorem 3. *$f \in A(G)$ and $f \in C^2(G)$ then $f \in Harm(G)$ $\left(\Re(f), \Im(f) \in Harm(G)\right)$*

Proof.

$$\triangle f \overset{\text{note}}{=} 4\frac{\partial}{\partial z}\overbrace{\left(\frac{\partial f}{\partial \bar{z}}\right)}^{=0(C-R)}$$

□

Theorem 4. *$f \in Harm(G)$ then $h(z) \in A(G)$ $(h : G \to \mathbb{R})$.*

Proof.

$$(h_z)_{\bar{z}} = h_{z\bar{z}} = \frac{1}{4}\triangle h \overset{\text{harmonic}}{=} 0$$

□

Definition 5. $u \in Harm_{\mathbb{R}}(G)$ v is conjugate to u if $v \in Harm_{\mathbb{R}}(G)$ and $u + iv \in A(G)$ or $v \in C^2(G)$

Theorem 5. *$u \in Harm_{\mathbb{R}}(\mathcal{D})$, $\mathcal{D}$ is a circle then there exists a conjugate v and v is unique up to a additive constant.*

Proof.

Uniqueness v_1, v_2 are conjugate to u, and G is domain. $f_1 = u + iv_1$, $f_2 = u + iv_2$

$$A(G) \ni f = f_1 - f_2 = i(v_1 - v_2)$$

$$\Re(f) = 0 \Rightarrow f = const \Rightarrow v_2 = v_1 + const$$

Existence We are looking for a $v \in C^2$ such that

$$\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases}$$

$$\begin{aligned} v(x,y) &= v(x,y) - v(x,y_0) + v(x,y_0) - v(x_0,y_0) + v(x_0,y_0) \\ &= \int_y^{y_0} v_y(x,t)\,dt + \int_x^{x_0} v_x(s,y_0)\,ds + v(x_0,y_0) \overset{C-R}{=} \int_{y_0}^{y} u_x(x,t)dt - \int_{x_0}^{x} u_y(s,y_0)\,ds + v(x_0,y_0) \end{aligned}$$

Which defines a function $v \in C^2(\mathcal{D})$ and v is conjugate to u.

□

Example 1. $u = x^2 - y^2 = \Re(z^2)$, $v = 2xy = \Im(z^2)$.

Example 2. $u = \log|z| = \frac{1}{2}\log(x^2 + y^2)$ is harmonic in $\mathbb{C}\setminus\{0\}$ $v = \arg(z) = \arctan\left(\frac{x}{y}\right)$.

2 Arc in $\mathbb{C}$

$I \subset \mathbb{R}$ is a section and $\gamma : I \to \mathbb{C}(\mathbb{R}^2)$, $\gamma \in C(I)$

Example 3. Line

$$\gamma(t) = (1-t)z + tw, \qquad 0 \le t \le q$$

Circle [2]

$$\gamma(t) = \cos(2\pi t) + i\sin(2\pi t)$$

Definition 6. $\gamma \in C^1$ if for all $t_0 \in I$

$$\exists \lim_{\substack{t \to t_0 \\ t \in I}} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

[2] If $\gamma(0) = \gamma(1)$ then the arc is called a **closed arc**