Complex Function Theory

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1 Symmetry principle

As we defined previously, $\Theta^* = \{s : \bar{z} \in \omega\}$, $\Theta = \Theta^*$, $\Theta_\pm = \Omega \cap \mathbb{C}_\pm$ and $I = \mathbb{R} \cap \Theta$.

Theorem 1. *Let $v \in Harm(\Omega_+ \cap C(\Omega_+ \cup I)$ and $v|_I = 0$, then*

$$\tilde{v}(z) = \begin{cases} v(z) & z \in \Omega_+ \\ 0 & z \in I \\ -v(\bar{z}) & z \in \Omega_- \end{cases}$$

is harmonic in $\mathbb{C}$.

Theorem 2. *Let $f \in Hol(\Omega_+)$ if for $z \to 0$, $\Im f(z) \to 0$ then f has a continuation $\tilde{f} \in Hol(\Omega)$ that satisfies $\tilde{f}(z) = \overline{f(\bar{z})}$for all $z \in \Omega$.*

Proof. It is sufficient to prove in the case that Ω is a circle. Let $f = u + iv$ that satisfies the above conditions, then v has a harmonic continuation in all of Ω and in Ω there exists a harmonic conjugate u to v. $u + iv \in Hol(\Omega) \Rightarrow$ there exists a continuation of f in Ω, proving the first part.
For the second we will check if $u(\bar{z}) \overset{??}{=} u(z)$. We will define $U(z) = u(z) - u(\bar{z})$ this holds that $U \in Harm(\Omega)$. $U|_I = 0$ and $\frac{\partial U}{\partial x}|_I = 0$.

$$\frac{\partial U}{\partial y}|_I = \frac{\partial u}{\partial y}|_I + \frac{\partial u}{\partial y}|_I = 2\frac{\partial u}{\partial y}|_I \overset{C-R}{=} -2\frac{\partial v}{\partial x}|_I \overset{v=0}{=} 0$$

Then $\frac{\partial x}{\partial x} = \frac{\partial u}{\partial y} = 0$ on I

$$\Rightarrow \partial_z U = \frac{1}{2}\left(\partial_x - i\partial_y\right) U = 0$$

Then $U = const$ and $U|_I = 0 \Rightarrow U = 0 \Rightarrow u(z) = u(\bar{z})$ □

Theorem 3. *Let $\Omega^* = \Omega$ and $I = \Omega \cap C$ for some clircle C. If*

1. *$f \in Hol(\Omega_+)$*
2. *There exsits a clircle C' such that $dist(f(z), C') \to 0$, $dist(z, C) \to 0, z \in \Omega_+$*

Then f has an analytic continuation $\tilde{f}$ to Ω and $\tilde{f}\left((z^*)_C\right) = \left(f(z)^*\right)_{C'}$

Note 1. If $f \in C(\Omega_+ \cup I)$ and $f(I) \subset C'$ then (2) holds.

Proof. The proof is mainly graphic, The idea is to take two mobius transformations, the first φ transfers C to the real line and the second ψ transfers C' to the real line. Then if we take $g = \psi \circ f \circ \varphi^{-1}$, then $g \in Hol(\varphi(\Omega_+))$ then $\Im g(w) \to 0$ as $\Im w \to 0$ □

Example 1. 1. Let $f \in Hol(\mathbb{D})$ and $\left|f(z)\right| \xrightarrow{x \to 1} 1$ then

$$f(z) = C\prod_{j=1}^{n}\frac{z - z_j}{1 - z\bar{z}_j}$$

For $z_1, \dots, z_n \in \mathbb{D}, |C| = 1$.

Proof. A special case :$f \neq 0$ in $\mathbb{D}$, then by the symmetry principle, f has an analytic continuation in all of $\hat{\mathbb{C}} \Rightarrow f \equiv C$,$|C| = 1$.
In the general case, $\#\{\text{zeroes of } f\}$ Let $z_1, \dots, z_n \in \mathbb{D}$ be the zeroes. We will define

$$g(z) = f(z) \cdot \prod_{j=1}^{n}\frac{z - z_j}{1 - z\bar{z}_j}$$

- $g \in Hol(\mathbb{D})$
- g doesnt have zeroes.
- $\left|g(z)\right| \to 1$ as $z \to 1$.

Therefore by the special case, $g \equiv C$,$|C| = 1$ □

2. We will define the annulus $A(r, R) = \left\{r < |z| < R\right\}$ and

$$f : \overbrace{A(r_1, R_1)}^{:=A_1} \xrightarrow[onto]{1-1} \overbrace{A(r_2, R_2)}^{:=A_2}$$

is analytic, $f = Cz$ or $f = \frac{c}{z}$ and $\frac{R_2}{r_2} = \frac{R_1}{r_1}$

Proof. Before we begin, some definitions:

- $C_1^- = \left\{z : |z = r_1|\right\}$,$C_1^+ = \left\{z : |z = R_1|\right\}$
- $C_2^- = \left\{z : |z = r_1|\right\}$,$C_2^+ = \left\{z : |z = R_2|\right\}$
- γ is a closed arc in A_2 which goes around the whole annulus.
- l is an arc which doesn't intersect with the inverse image of γ and starts at z' and ends at z'' (to be defined later)

$z \to \partial A_1 \Rightarrow f(z) \to \partial A_2$ $(dist(z, \partial A_1) \to 0 \Rightarrow dist(f(z), \partial A_2) \to 0)$. We will assume that this is not true: $z_j \to \partial A_1 \Rightarrow f(z_j) \to w \in A_2$. $\delta < \frac{1}{2}dist(\partial A_1, \bar{u_z}$ ($\bar{u_z}$ is a neighboorhood of z). Thus for a big enough j, $dist(z_j, \partial A_1) < \delta \Rightarrow z_j \notin u_z$. If $z \to C_1^- \Rightarrow f(z) \to C_2^-$ or $f(z) \to C_2^+$ Assume that there exists $z_j', z_j'' \to C_1^-$ such that $f(z_j') \to C_2, f(z_j'') \to C_2^+$. $l \cap f^{-1}(\gamma) \neq \emptyset$

$$z \in l \Rightarrow r_1 < |z| < r_1 + \varepsilon \qquad r_1 + \varepsilon < \min\left\{|z| : z \in f^{-1}(\gamma)\right\}$$

which gives us that $f(l) \cap \gamma \neq \emptyset$, a contradiction!
We have arrived at a reduction to one of the following two cases: If $f : \mathbb{C}\backslash\{0\} \xrightarrow[onto]{1-1} \mathbb{C}\backslash\{0\}$, $\partial C^* = \{0, \infty\}$

(a) $z \to 0 \Rightarrow f(z) \to 0$, $z \to \infty \Rightarrow f(z) \to \infty$

(b) $z \to \infty \Rightarrow f(z) \to 0$, $z \to 0 \Rightarrow f(z) \to \infty$

For each case we ahve

(a) f is analytic also in $z = 0 \Rightarrow f \in Hol(\mathbb{C})$ and is injective, therefore $f(z) = Cz + B$ and since $f(0) = 0$ we have $f(z) = Cz$. In addition we have $cr_1 = r_2$, $cR_1 = R_2$.

(b) $\frac{1}{f} : C^* \xrightarrow[onto]{1-1} C^*$ thus we have the first case, $f(z) = \frac{C}{z}$ and $\frac{C}{r_1} = R_2$, $\frac{C}{R_1} = r_2$

Either way, we have the required result that

$$\frac{R_2}{r_2} = \frac{R_1}{r_1}$$

$\square$