Complex Function Theory

Mikhail Sodin
Arazim ©

January 7, 2016

1 Poisson integral

In the last lesson we defined the integral

$$P(z) = \frac{1-|z|}{|1-z|^2} = \frac{1-r^2}{1-2r\cos\theta+r^2}, \qquad re^{i\theta} = z = p_r(\theta)$$

Let $H \in C(\mathbb{T})$, $h(\varphi) = H(e^{i\varphi})$, $h \in C([-\pi,\pi])$, $h(\varphi+2\pi) = h(\varphi)$

$$P_H(z) = \frac{1}{2\pi}\int_0 2\pi H(e^{i\varphi})\frac{1-|z|^2}{|e^{i\varphi}|^2}d\varphi = \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)p_r(\theta-\varphi)d\varphi$$

1. $P_H \in Harm(\mathbb{D})$
2. $P_{cH} = cP_H$, $P_{H_1+H_2} = P_{H_1} + P_{H_2}$
3. $H \geq 0 \Rightarrow P_H \geq 0$.
4. $P_{\mathbb{1}} = \mathbb{1}$
5. $M \geq H \geq m \Rightarrow \forall z \in \mathbb{D}, m \leq P_H(z) \leq M$

Theorem 1 (Schwartz theorem)**.** *If $H \in C(\mathbb{T})$ then $P_H \in C(\bar{\mathbb{D}})$:*

$$P_H(z) \xrightarrow[z\in\mathbb{D}]{z\to e^{i\varphi}} H(e^{i\varphi})$$

Proof. We will set $e^{\varphi_0} \in \mathbb{T}$!!!!!!!! □

Corollary 1 (Schwartz)**.** *Let $G \subset \mathbb{C}$ be a domain and $u : G \to \mathbb{R}^1$ be continuous. Assume that for all $z_0 \in G$ there exists a $\rho = \rho(z_0) > 0$ such that for all $0 < r < \rho$*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\varphi})d\varphi$$

Then $u \in Harm(G)$.

Proof. Let $G \supset \bar{\mathbb{D}} = \{z : |z-z_0| < \rho\}$. u_1 is a solution to the Dirichlet problem in D

- $u_1 = u$ in $\partial\mathfrak{D}$
- $\triangle u_1 = 0$ in $\mathfrak{D}$
- $u_1 \in C(\bar{\mathfrak{D}}$

Thus, if we define $v = u - u_1$, by the Schwartz theorem, such a u_1 exists and $v|_{\partial\mathfrak{D}} = 0$. u_1, u doesn't have a local maximum or minimum in $\mathfrak{D} \Rightarrow v = u - u_1$ also has no local minima or maxima, by the maximum principle, $\Rightarrow \min_{\partial\mathfrak{D}} v(z) \leq \max_{\partial\mathfrak{D}} v$ in all of $\mathfrak{D} \Rightarrow v = 0 \Rightarrow u_1 = u$ □

1.1 Poisson integral for $\mathbb{C}_+ = \left\{\Im(Z) > 0\right\}$

Let $z = x + iy, y > 0$

$$P(z) = \frac{y}{\pi}\frac{1}{x^2+y^2} = -\pi\Im\frac{1}{z}$$

- Harmonic in $\mathbb{C}_+$
- $h \in C(\mathbb{R})$ is bounded $\Rightarrow$

$$P_h(z) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{h(t)dt}{(x-t)^2+y^2} = \int_{-\infty}^{\infty} h(t)p(x-t,y)dt$$

Some properties for this integral:

1. $P_h \in Harm(\mathbb{C}_+)$
2. $P_h(z) \to h(t), \mathbb{C}_) \ni z \to t$
3. If $v\ inHarm(\mathbb{C}_+) \cap C(\bar{\mathbb{C}_+})$, bounded and $v|_{\mathbb{R}}$, then $v = P_h$ in $\mathbb{C}_+$.

The proofs for this are simple, notice that we had to add the property that h is bounded. We can use the Möbius transformation $\frac{z-1}{z+1}$ in order to send the half plane to the circle.

2 Symmetry principle (reflection)

Let $\Theta \subset \mathbb{C}$ be a domain, we will define $\Theta^* = \{z : \bar{z} \in \Theta\}$. Θ is symmetric around $\mathbb{R}$ of $\Theta^* = \Theta$, $\Theta_+ = \Theta \cap \mathbb{C}_+$, $I = \Theta \cap \mathbb{R}$

Theorem 2 (Harmonic version)**.** *For a Θ such that $\Theta = \Theta^*$. Let $v \in Harm(\Theta_+) \cap C(\Theta_+ \cup I)$, $v|_I = 0$ then !!!!!!!!!!*