Complex Function Theory

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1 Harmonic functions

Definition 1. Let $G \subset \mathbb{C}$, we will define the harmonic functions as:

$$Harm(G) = \left\{ f : G \to \mathbb{R}, f \in C^2(G), \triangle f = 0 \right\}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y}\right) \qquad \triangle f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

1. $$Hol(G) \subset Harm_{\mathbb{C}}(G) \qquad f \in Hol(G) \Rightarrow \Re f, \Im f \in Harm(G)$$

2. $$u \in Harm_{\mathbb{C}}(G) \Rightarrow \partial_z u \in Hol(G) \qquad 0 = \frac{\partial^2 u}{\partial \bar{z} \partial z} = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial u}{\partial z}\right) = 0 \Rightarrow \frac{\partial u}{\partial z} \in Hol(G)$$

3. If G is smply connected, $u \in Harm(G) \Rightarrow \exists f \in Hol(G)$

   Proof. $g = 2\partial_z u \in Hol(G) = u_x - iu_y$. There exists an antiderivative, $f \in Hol(G), f(z_0) = u_0, f' = g$ (we chose the integration constant such that $f(z_0) = u_0$). For $f = u_1 + iv_1$ we will check if $u_1 \overset{???}{=} u$

   $$f' = \partial_x u_1 + i\partial_x v_1 = \partial_x u_1 - i\partial_y u_1 \overset{C-R}{\Rightarrow} \begin{cases} \partial_x u = \partial_x u_1 \\ \partial_y u = \partial_y u_1 \\ u_1(z_0) = u_(z_0) \end{cases} \Rightarrow u_1 = u$$

   □

4. $$u \in Harm(G) \Rightarrow \forall z \in G \exists u_z \subset G . \exists f \in Hol(u_z) . u \Re f \in C^\infty(G)$$

5. $Harm(G) \subset C^\infty(G)$

6. $$u \in Harm(G) \Rightarrow \forall z_0 \in G \forall \rho < dist(z_0, \partial G) . u(z_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(z_0 + \rho e^{i\theta}\right) d\theta$$

7. (Uniqueness theorem) Let $u \in Harm(G)$, if there exists an open $E \subset G$ such that $u|_E = 0$ then $u = 0$ on all of G.

Proof. We will define $\mathcal{O} = \{z \in G : u(z) = 0\}$, then we have that $\emptyset \neq int(\mathcal{O}) \stackrel{??}{=} G$. Assume that $z_0 \in \partial\mathcal{O} \cap G$ and $\mathfrak{D} = \mathfrak{D}_{z_0} \subset G$. Then by (4), there exists a $f \in Hol(\mathfrak{D})$ such that $u = \Re(F)$. Since $z \in \mathfrak{D} \cap \mathcal{O} \Rightarrow \Re(f) = 0 \Rightarrow f = iC$ in $\mathfrak{D} \cap \mathcal{O}$ and from the uniqueness of analytic functions we have $f = iC$ in all of $\mathfrak{D} \Rightarrow u = 0$ in all of $\mathfrak{D}$ and we have reached our contradiction (since $z_0 \in \partial O$). □

8. (Maximum principle) Let $G \subset \mathbb{C}$ be a bounded domain, and $u \neq const, u \in Harm(G) \cap C(\bar{G})$. Then for all $z \in G$,

$$u(z) < \max_{\bar{G}} u \qquad \left(\max_{\bar{G}} u = \max_{\partial G} u\right)$$

Proof.

$$K = \left\{z \in : u(z) = \max_{\bar{G}} u\right\}$$

- From (6) we know that K is open.
- $G \backslash K$ is open (u is continuous).

$\Rightarrow K = G \Rightarrow u = const$
$\Rightarrow K = \emptyset \Rightarrow \max_{\bar{G}} u > u$ □

2 Poisson integral

Let $h \in C(\partial G)$ we are looking for $u \in Harm(G) \cap C(\bar{G})$ such that $u|_{\partial G} = h$ (Dirichlet problem)

Uniqueness $u_1, u_2 \in Harm(G) \cap C(\bar{G})$ and $u_1 = u_2 = h$ in ∂G. Thus, by the maximum principle, $u_1 - u_2 = 0$ in $\partial G \Rightarrow u_1 = u_2$ in G

Theorem 1. *$u \in Harm(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ then for all $z \in \mathbb{D}$ we have*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u \left(e^{i\varphi} \frac{1 - |z|^2}{\left|z - e^{i\varphi}\right|^2} d\varphi\right)$$

Proof. $u \in Harm(\bar{\mathbb{D}}$ If not then $u_t(z) = u(tz), t < 1, u_t \in Harm(\bar{\mathbb{D}})$ and we prove for u_t as $t \uparrow 1$).
In the special case, $z = 0$

$$u(0) = \frac{1}{2} \int_0^{2\pi} u(e^{i\varphi}) d\varphi$$

In the general case: We will set $z \in \mathbb{D}$ and define $v(\zeta) = u(\frac{\zeta + z}{1 + \bar{z}}$ then $u \in Harm(\mathbb{D} \cap C(\bar{\mathbb{D}})$.

$$u(z) = v(0) = \frac{1}{2\pi} \int_0^{2\pi} v\left(e^{i\psi}\right) d\psi = \frac{1}{2\pi} \int_0^{2\pi} u\left(\frac{e^{i\psi} + z}{1 + e^{i\psi}\bar{z}}\right) d\psi$$

$$\left[e^{i\varphi} = \frac{e^{i\psi} + z}{1 + e^{i\psi}\bar{z}}, e^{i\psi} = \frac{e^{i\varphi} - z}{1 - e^{i\varphi}\bar{z}}\right]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\varphi}) \frac{d\psi}{d\varphi} d\varphi = ie^{i\psi} \frac{d\psi}{d\varphi} = \frac{ie^{i}}{den}$$

!!!!!!!!!!!!!!!!!!!!!!!! □

Corollary 1. *Let $f \in Hol(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) \frac{1 - |z|^2}{\left|z - e^{i\varphi}\right|^2} d\varphi$$

Corollary 2 (Schwartz equation)**.** *Let $f \in Hol(\mathbb{D}) \cap C(\bar{\mathbb{D}})$, $u = \Re f$ then*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\frac{e^{\varphi}+z}{e^{i\varphi}-z}d\varphi + i\Im f(0)$$

proof of Schwartz equation. We will define $h \in Hol(\mathbb{D})$ to be the integral in the Schwartz equation. $\Re f = u$ and

$$\Re h = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\frac{e^{\varphi}+z}{e^{i\varphi}-z}d\varphi$$

Thus $Poisson = u \Rightarrow f = h + iC$.

$$u(0) + i\Im(f(0)) = f(0) = \overbrace{h(0)}^{=u(0)} + iC \Rightarrow C = \Im(f(0))$$

□

2.0.1 Poisson kernel

We will definet the Poisson kernel

$$\frac{1 - |z|^2}{\left|z - e^{i\varphi 2}\right|}$$

!!!!!!!!!!!!! For $z = re^{i\theta}$

$$\frac{1-|z|^2}{\left|e^{i\varphi}-z\right|^2} = \Re\left(\frac{e^{i\varphi}+z}{e^{i\varphi - z}}\right) = \frac{1-r^2}{1 - 2\cdot\cos(\theta\cdot\varphi) + r^2}$$

For $R\mathbb{D}$

$$\frac{1 - \left|{}^{z}/_{R}\right|^2}{\left|e^{i\varphi} - {}^{z}/_{R}\right|^2} = \frac{R^2 - |z|^2}{\left|Re^{i\varphi} - z\right|^2}$$

And the Schwartz kernel for all $|z| < R$

$$\frac{Re^{i\varphi}+\zeta}{Re^{i\varphi}-\zeta}$$

$$p_r) := P(z) = \frac{1-|z|^2}{|1-z|^2} = \frac{1 - z62}{1 - 2r\cos\theta + r^2}$$

- $p_r \geq 0$
- $p_r(-\theta) = p_r(\theta)$
- $z \to p(z)$ is harmonic and

$$\frac{1}{2\pi}\int_0^{2\pi} p_r(\theta)d\theta = 0 (= p(0) = 1)$$

- The maximum over $\lambda \leq |\theta| \leq \pi$ as $r \uparrow 1$ is $p_r \to 0$.

$H : \Pi \to \mathbb{R}$ ($\Pi = \partial\mathbb{D}$), $H(e^{i\varphi}) = h(\varphi)$ for $h : [-\pi, \pi] \to \mathbb{R}$ $H \in C(\Pi)$

$$P_H(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi)p_r(\theta - \varphi)d\varphi$$

Poisson integral

1. $P_H \in Harm(\mathbb{D})$
2. $P_{cH} = cP_H$ and $P_{H_1+H_2} = P_{H_1} + P_{H_2}$ (Linear funcitional).
3. $H \geq 0 \Rightarrow P_H \geq 0$
4. $P_{\nvDash} = \nvDash$!!!!!!!!!!