Complex Function Theory

Mikhail Sodin
Arazim ©

January 5, 2016

1 Simpy connected domains

Definition 1. $G \subset \mathbb{C}$ is a simply connected domain if every arc $\gamma : [0,1] \to G$ such that $\gamma(0) = \gamma(1)$ is contractible in G.

Corollary 1. *1. $G \subset \mathbb{C}$ is a simply connected domain $\Rightarrow$*

$$\forall f \in Hol(G).\forall \gamma : I \to G.\gamma(1) = \gamma(0). \int_\gamma f(\zeta)d\zeta = 0$$

Which is equivalent to the fact that $\int_\gamma f$ depends only on its edges.

2. If $G \subset \mathbb{C}$ is a simply connected domain, then

$$\forall f \in Hol(G).\exists F : F' = f \; \big(F \in Hol(G)\big)$$

3. $G \subset \mathbb{C}$ is a simply connected domain. Let $f \in Hol(G), f \neq 0$ then there exists $h, g \in Hol(G)$ such that

$$e^g = f \,(g \text{ is a branch of } \log f) \qquad h^2 = f \,(h \text{ is a branch of } \sqrt{f})$$

Proof. We will define $e^c := f(z_0)$ and $g(z) = \int_\gamma \frac{f'}{f}(\zeta)d\zeta + c$. Then $g \in Hol(G)$ and $g' = \frac{f'}{f}$

$$\Big(fe^{-g}\Big)' = f'e^{-g} - fg'e^{-g} = \big(f' - fg'\big)\, e^{-g} = 0 \Rightarrow fe^{-g} = const$$

$$f(z_0)e^{-g(z_0)} = f(z_0)e^{-c} = 1 \Rightarrow fe^{-g} = 0 \Rightarrow e^g = f$$

□

Theorem 1 (Riemann theorem)**.** *Let $G \subsetneq \mathbb{C}$ be a simply connected domain and $z_0 \in G$ then there exists a biholomorphic map:*

$$f : (G, z_0) \xrightarrow[onto]{1-1} (\mathbb{D}, 0) \text{ and } f'(z_0) > 0 \,(\arg f'(z_0) = 0)\,(f(z_0) = 0)$$

Proof.

$$g := f \circ f_1^{-1} : (\mathbb{D}, 0) \to (\mathbb{D}, 0) \overset{Schwartz}{\Rightarrow} \big|g'(0)\big| \le 1 \Rightarrow \big|f'(z_0)\big| \le \big|f_1'(z_0)\big|$$

If we define $g_1 = f_1 \circ f^{-1} : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$, in a similiar fashion we ahve that $\big|f'(z_0)\big| \ge \big|f_1'(z_0)\big|$. Then $\big|g'(0)\big| = 1$ and again from Schwartz's lemma, $g(w) = e^{i\varphi}w$, therefore $w = f_1(z), f(f_1(w)) = e^{i\varphi}$!!!!!! □

Theorem 2. *Let $G \subsetneq \mathbb{C}$ be a domain. The following are equivalent:*

1. G is simply connected.

2. *Got all $f \in Hol(G)$ there exists an antiderivative.*

3. *For all $z \in \mathbb{C} \backslash G$ there exists a branch of $\zeta \to \log(\zeta - z)$ in G.*

4. *For every closed arc $g : I \to G$, and for all $z \in \mathbb{C} \backslash G$, $ind_\gamma = 0$.*

5. *$\mathbb{C} \backslash G$ is connected.*[1]

Proof. We have already shown that (1) $\Rightarrow$ (2) $\Rightarrow$ (3).
(3) $\Rightarrow$ (4)

$$ind_\gamma(z) = \frac{1}{2\pi} \triangle_\gamma \arg(\zeta - z) = 0$$

(4) $\Rightarrow$ (3)

$$ind_\gamma(z) = 0 \Rightarrow \int_\Gamma \frac{d\zeta}{\zeta - z}$$

Depends only on the edges of $\Gamma : I \to G$ (Cauchy theorem with index).

$$w \mapsto \int_{w_0}^{2} \frac{d\zeta}{\zeta - z}$$

Is analytic in G!!!!!!!!!!!!! □

[1] X is connected if every continuous function $h : X \to \mathbb{Z}$ is constant.