Complex Function Theory

Mikhail Sodin
Arazim ©

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Theorem 1 (Caucy Theorem)**.** *Let $f \in Hol(G)$ and $\gamma : I \to G$ be an arc, closed and piecewise C^1. If for all $z \in \mathbb{C} \backslash G : ind_\gamma(z) = 0$ then*

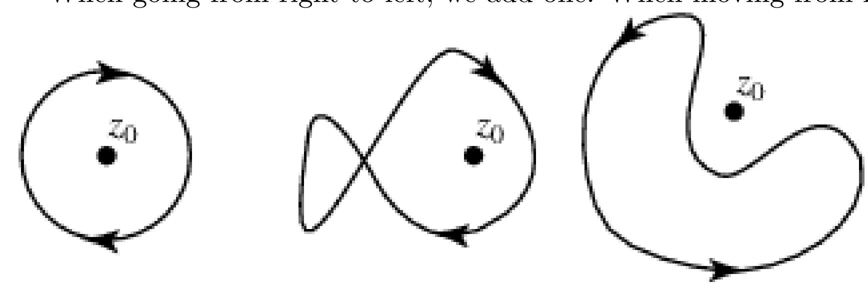
$$\int_\gamma f(\zeta) d\zeta = 0$$

Corollary 1. *If $z \in G$ then*

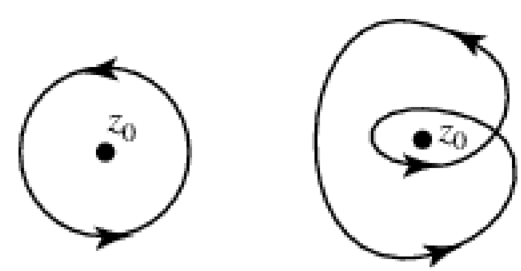
$$f(z) \cdot ind_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

Note 1 (Cycle)*.* A finite sum of sums $\gamma = \sum_{finite} n_j \gamma_j$ where each γ_j is a closed arc and $n_j \in \mathbb{N}$.

When going from right to left, we add one. When moving from left to right, remove one.



$\mathrm{Ind}_\gamma(z_0) = -1$ $\quad$ $\mathrm{Ind}_\gamma(z_0) = -1$ $\quad$ $\mathrm{Ind}_\gamma(z_0) = 0$



$\mathrm{Ind}_\gamma(z_0) = +1$ $\quad$ $\mathrm{Ind}_\gamma(z_0) = +2$

$$ind_\gamma(z) := \sum n_j ind_{\gamma_j}(z)$$

$$\int_\gamma f(\zeta) d\zeta = \sum n_j \int_{\gamma_j} f(\zeta) d\zeta$$

Theorem 2. *γ is a piecewise C^1 arc and $f \in C(\gamma)$, then*

$$F(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

and $F \in Hol(\mathbb{C}\backslash\gamma), F(\infty) = 0.$

0.1 Standard assumptions

The folowing are referred to as the standard assumptions

1. $f \in Hol(\mathfrak{D}$.
2. γ enters $\mathfrak{D}$ at ζ_1 and γ exits $\mathfrak{D}$ at ζ_2.
3. $\gamma 0 = \gamma \cap \mathfrak{D}$ divides $\mathfrak{D}$ into $\mathfrak{D}_-$ and $\mathfrak{D}_+$ ($\mathfrak{D}_+$ is on the left of γ_0).
4. $\mathfrak{D} \cap (\gamma\backslash\gamma_0) = \emptyset$

Defining

$$F_\pm = F\Big|_{\mathfrak{D}_\pm}$$

And γ_- is the right side of $\mathfrak{D}$ and γ_+ is the left side of $\mathfrak{D}$.

Note 2. $F_\pm$ has a analytic continuation on $\mathfrak{D}$.

$$F_+(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

${}^{z\in\mathfrak{D}_+}12\pi i \left[\int_{\gamma\backslash\gamma_0} + \int_{\gamma_0}\right] \frac{f(\zeta)}{\zeta - z}d\zeta$and the RHS is analytic in $\mathbb{C}$. By Cauchy we have

$$\int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}d\zeta = 0$$

Theorem 3.

$$F_+(z) - F_-(z) = f(z)$$

Proof.

$$F_+(z) - F_-(z) = \frac{1}{2\pi i}\left[\int_{\gamma_-} - \int_{\gamma_+}\right]\frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\partial\mathfrak{D}}\frac{f(\zeta)}{\zeta - z}d\zeta = f(z)$$

□

Theorem 4. *Let* $\gamma : [0,1] \to \mathbb{C}$ *be a simple closed arc, then* $\Omega_0 \cup \Omega_1 = \mathbb{C}\backslash\gamma$, $\partial\Omega_0 = \partial\Omega_1 = \gamma$ *are two disjoint domain were* Ω_1 *is unbounded (the "outside") and* Ω_0 *is bounded (the "Inside").*

proof for C^1 *arcs.* Since γ is compact, we can write $\gamma \subset \cup_{finite}\mathfrak{D}_j =: u$. and $\mathbb{C}\backslash\gamma = \cup_{0\le j\le N}\Omega_j$ is open and $N < \infty$ (Ω_j are the components).

- If $N \le 2$ then there exists an arc in u that begins in $\mathfrak{D}_j^\pm\backslash\gamma_-$ and ends at $\mathfrak{D}_k\backslash$

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1 Homotopy

Let $\gamma_0 : [0,1] \to \mathbb{C}$, $\gamma_1 : [0,1] \to \mathbb{C}$, $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$. $Q = [0,1] \times [0,1]$!!!

Definition 1. $G \subset \mathbb{C}$ is a domain. $\gamma_0, \gamma_1 : [0,1] \to G$. $\gamma_0(0) = \gamma_1(0)$ $\gamma_0(1) = \gamma_1(1)$. We will say that $\gamma_0 \sim_G \gamma_1$ if there exists a homotopy $\gamma : Q \to G$ such that $\gamma(t,0) = \gamma_0(t)$ and $\gamma(t,1) = \gamma_1(t)$

Theorem 5 (Homotopical form)**.** *If $f \in Hol(G)$ and $\gamma_0 \sim_G \gamma_1$, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Corollary 2. *$\gamma : [0,1] \to G$,is $\gamma \sim_G id$ ($id : [0,1] \to \{a\}$) meaning that γ is contractible then*

$$\int_{\gamma} f(\zeta) d\zeta = 0$$

Lemma 1 (Continuous logarithm)**.** *$h \in C(Q), h : Q \to \mathbb{C} \setminus \{0\}$ then there exists a $\phi \in C(Q)$ such that $h = e^{\phi}$.*

Lemma 2. *If $\gamma_0 \sim_G \gamma_1$, then for all $z \in \mathbb{C} \backslash G$*

$$ind_{\gamma_1 - \gamma_0}(z) = 0$$

Proof. $\gamma_0 \sim_G \gamma_1 \Rightarrow \exists \gamma : Q \to G$ where γ is a homotopy from $\gamma_0 \to \gamma_1$. If $z \in \mathbb{C} \backslash G \Rightarrow \gamma(w) \neq z$ fr some $w = (t,s) \in Q$

$$h(w) = \gamma(w) - z \neq 0 \Rightarrow \exists \phi \in C(Q) : \gamma(w) - e^{\phi(w)}$$

Let $\zeta \in \gamma_1 : \zeta - z = \gamma_1(w) - z = e^{\phi(w)}$, for $w = (t,1)$. We have

$$ind_{\gamma_1 - \gamma_0}(z) = \frac{1}{2\pi} \Big[\triangle \arg_{\zeta \in \gamma_1}(\zeta - z) - \triangle \arg_{\zeta \in \gamma_0}(\zeta - z) \Big] = \frac{1}{2\pi} \Im \Big[\big(\phi(1,1) - \phi(0,1)\big) - \big(\phi(1,0) - \phi(0,0)\big) \Big] = 0$$

Defining $\gamma_s = \gamma(t,s)$. The function $s \mapsto ind_{\gamma_s - \gamma_0}(z)$ is continuous on s since in a similar action to before, it is equal to the imaginary part of four items. Since this is a continuous function we have that $ind_{\gamma_s - \gamma_0}(z) = const$. ☐

Definition 2. LEt $G \subset \mathbb{C}$, it is called a simply-connected domain if every closed arc $\gamma : [0,1] \to G$ is contractible.

Example 1.

1. $\mathfrak{D}(z,\rho) = \{\zeta : |\zeta - z| < \rho\}$
2. G is a convex domain.

   Proof. A homotopy for a convex domain is $\gamma(t,s) := (1-s)\gamma_0(t) + s\gamma_1(t)$ and $\gamma \in C(Q)$. $\gamma_0(0) = \gamma_1(0) = a$,$\gamma_0(1) = \gamma_1(1) = b$. ☐

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