Complex Function Theory

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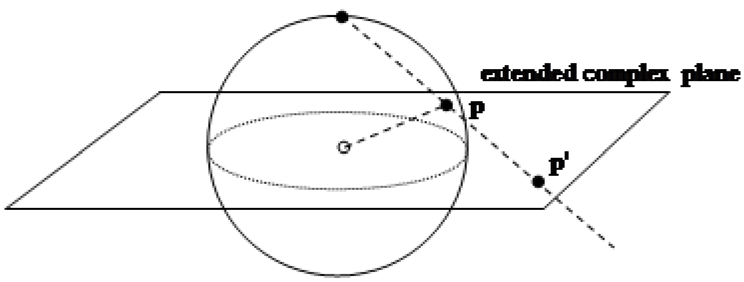
1 Riemman sphere



The transformation $z \mapsto (\xi, \eta\lambda)$. Is a transformation from a point on the complex plane to a point on the upper hemisphere. This is achieved by drawing a line from the north pole of the sphere to the point z and finding the point of intersection.
Line: $\big(tx, ty, (1-t)\big)$.

$$t^2x^2 + t^2y^2 + (1-t)^2 = 1$$

$$t^2\left(x^2 + y^2 + 1\right) - 2t = 0$$

$$t = \frac{2}{x^2 + y^2 + 1} = \frac{2}{|z|^2 + 1}$$

$$\xi = \frac{2x}{|z|^2 + 1} \qquad \frac{2y}{|z|^2 + 1} \qquad \lambda = \frac{|z|^2 - 1}{|z|^2 + 1}$$

An inverse function $(\xi, \eta, \lambda) \mapsto (x, y)$ can be shown as:

$$x = \frac{\xi}{1-\lambda} \qquad y = \frac{\eta}{1-\lambda} z = \frac{\xi + i\eta}{1-\lambda}$$

We have found a transformation from the complex plane to the Riemann sphere!

1.1 Chordal distance

$$\rho(z_1, z_2) := \sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\lambda_1 - \lambda_2)^2} = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}$$

Now we can find the limit $z_2 \to \infty$

$$\rho(z, \infty) := \lim_{w \to \infty} \rho(z, w) = \frac{1}{\sqrt{1 + |z|^2}}$$

2 Differentiability - ($\mathbb{C}$)

$G \subset \mathbb{C}$ is an open set. $z \in G, f : G \to \mathbb{C}$

Definition 1. f is differentiable in $\mathbb{C}$ at z if the following limit exists:

$$f;(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Example 1.
- $f(z) = e \qquad f'(z) = 0$
- $f(z) = z) \qquad f'(z) = 1$
- $f(z) = z^2 \qquad f'(z) = 2z$

If f is differentiable in $\mathbb{C}$ then f is differentiable in $\mathbb{C}$. As we learned in Calculus 2,the following rules still apply in $\mathbb{C}$:

- $(cf)' = cf'$
- $(f \pm g)\, f' \pm g'$
- $(f \cdot g)' = f'g + fg'$
- $\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \qquad \big(g(z) \neq 0\big)$

Claim 1 (Chain rule). g is differentiable at z_0 and f is differentiable at $w = g(z_0)$then

$$(f \circ g)'(z_0) = f'\left(g(z_0)\right) g'(z_0)$$

Proof. If $g'(z_0) \neq 0, g(z_0 + h) - g(z_0) \neq 0$ for a small enough h then

$$\frac{f\left(g(z_0+h) - f\left(g(z_0)\right)\right.}{h} = \frac{f(g(z_0+h) - f(g(z_0))}{g(z_0+h) - g(z)} \cdot \frac{g(z+h) - g(z)}{h} \xrightarrow[h \to 0]{f} {}' (g(z))g'(z)$$

Else, if $g'(z) = 0$: Let $w \in \mathbb{C}$ such that $|w - w_0| < \varepsilon$ then since f is differentiable at w_0 we have $\big|f(w) - f(w_0)\big| \leq C|w - w_0|$ and if $w = g(z_h)$ we have

$$\left|\frac{f\left(g(z_0+h) - f\left(g(z_0)\right)\right.}{h}\right| \leq C \cdot \overbrace{\left|\frac{g(z+h) - g(z)}{h}\right|}^{\to 0}$$

Thus, $(f \circ g)'(z) = 0$ ☐

In order to show the difference between differentiability in Calculus 2 and here we will look at the conjugacy function. $f(z) = \bar{z} = x - iy$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix}$$

In $\mathbb{R}^2$ this transformation is in C^∞, However in $\mathbb{C}$...

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \bar{z}}{h} = \frac{\bar{h}}{h}$$

As we can see here, there is no limit as $h \to 0$!.
When is a function $f(z) = u(x, y) + iv(x, y)$ differentiable in $\mathbb{C}$?

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

The difference between $\mathbb{C}$ and $\mathbb{R}^2$ is shown in the following equations representing linear transformations in $\mathbb{R}$ and $\mathbb{C}$ respectively:

$$(\mathbb{C}) \qquad f(z) = az + b\bar{z} \qquad a, b \in \mathbb{C}$$

$$\left(\mathbb{R}^2\right) \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{pmatrix}$$

We can see that f is differentiable in $\mathbb{C} \Leftrightarrow b = 0$ (since $\bar{z}$ is differentiable and z is not) $\Leftrightarrow f(z) = az = ax + iay$.

$$f(z) = (\alpha x + \beta y) + i\left(\gamma x + \delta y\right) = \left(\alpha + i\gamma\right)x + \left(\beta + i\delta\right)y \Rightarrow \beta = -\gamma \qquad \delta = \alpha$$

Corollary 1. *A linear transformation is differentiable in $\mathbb{C}$ only if it is anti-symmetric!*

These lead us to the next lesson where will show the Cauchy-Riemman equations (also known as Euler-D'Alambert).