Complex Function Theory

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1 Geometric principles

1.1 Meromorphic principles

Let $G \subset \mathbb{C}$ be a domain. We say that $f \in M(G)$ if

$$\forall a \in G . \exists u_a^* \subset G : f \in Hol\left(u_a^*\right)$$

and a is a removable sinularity or a pole.

$$\infty \in M(G)$$

The mermophic function form a field, meaning that:

$$f_1, f_2 \in M(G) \Rightarrow f_1 + f_2, f_1 \cdot f_2, \frac{f_1}{f_2} \in M(G)$$

Theorem 1. [1]

$$\forall f \in M(G) . \exists f_1, f_2 \in H(G) . f = \frac{f_1}{f_2}$$

Theorem 2 (Argument principle)**.** *Let $G_1 \subset\subset G \subset \mathbb{C}(\bar{G}_1 \subset G)$ be a good domain, $\Gamma = \partial G_1$ and $f|_\Gamma \neq 0, \infty$. We will define*

$$N_f(G_1) = \#\left\{a \in G_1 : f(a) = 0\right\}$$

$$P_f(G_1) = \#\left\{a \in G_1 : f(a) = \infty\right\}$$

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'}{f} dz = N_f(G_1) - p_f(G_1)$$

Proof. According to the residue theorem:

$$\frac{1}{2\pi i} \cdot \int_\Gamma \frac{f'}{f} dz = \sum_{a \in G_1} res_a \left(\frac{f'}{f}\right) = N_f(G_1) - P_f(G_1)$$

!!!!!! □

A naive understanding of the first value is:
If we define $\Gamma : z = \gamma(t), t \in I = [a,b]$. $\gamma(a) = \gamma(b)$.

$$\int_\Gamma \frac{f'}{f}(z)\, dz = \int_I \frac{f'}{f}\left(\gamma(t)\right) \dot{\gamma}(t)\, dt = \log\left(f\left(\gamma(b)\right)\right) - \log\left(f\left(\gamma(a)\right)\right) = \triangle_\Gamma \log f = i \triangle_\Gamma \arg f$$

Lemma 1 (Continuous log)**.** $I = [a,b], h : I \to \mathbb{C} \setminus \{0\}, h \in C(I)$. *Then*

[1] No proof is given here, we will show this in the course complex function theory 2

1. *THere exists a* $\varphi \in C(I)$ *such that* $h = e^{y} \in I$.

2. *If* $\varphi_1, \varphi_2 \in C(I), e^{\varphi_!} = e^{\varphi_2} \Rightarrow \varphi_1 - \varphi_2 \equiv 2\pi i k, k \in \mathbb{Z}$

Proof. 1. **Special case:** If $h(I) \subset \Pi$ where Π is a half plane, and in half plane we can always choose a branch of the logarithm.
General case: $\forall t \in I, h(t) \neq 0 \Rightarrow \exists u_t \subset I \subset \Pi_t$. Using the Heine-Borel theorem from calculus 1 we get that

$$I \subset \bigcup_{\text{finite}} u_j, u_{j+1} \cap u_j \neq \emptyset$$

$$\forall j. \exists \varphi_j \in C(u_j), h = e^{\varphi_j}, t_j \in u_j \cap u_{j+1}$$

Choosing a φ_j such that $\varphi_{j+1}(t_j) = \varphi_j(t_j)$, by (2) we have that $\varphi_{j+1} = \varphi_j$ in $u_{j+1} \cap u_j$, $t \in u_j, \varphi \in C(I)$ In $u_j \cap u_{j+1}$ we have

$$\begin{cases} e^{\varphi_j} = h \\ e^{\varphi_{j+1}} = h \end{cases} \Rightarrow \tilde{\varphi_{j+1}} = \varphi_{j+1} + 2\pi i k, \tilde{\varphi_{j+1}} = \varphi_j, u_j \cap u_{j+1}$$

2.
$$e^{\varphi_1} = e^{\varphi_2} \Rightarrow e^{\varphi_1 - \varphi_2} = 1 \Rightarrow \forall t \in I : (\varphi_1 - \varphi_2)(t) \in 2\pi i \mathbb{Z}$$

Since these are discrete points we arrive at $\varphi_1 - \varphi_2 \equiv 2\pi i k, k \in \mathbb{Z}$

□

Theorem 3 (Rouche). *$f, g \in H(G)$, $G_1 \subset\subset G$ is a good domain. $\Gamma = \partial G_1$. If $|g| \underset{<}{\Gamma} |f|$ then $N_{f+g}(G_1) = N_f(G_1)$*

Proof.

$$N_{f+g}(G_1) = \frac{1}{2\pi} \triangle_\Gamma \arg(f+g) = \frac{1}{2\pi} \triangle_\Gamma \arg f \cdot \left(1 + \frac{g}{f}\right) = \overbrace{\frac{1}{2\pi} \triangle_\Gamma \arg f}^{N_f(G_1)} + \overbrace{\frac{1}{2\pi} \triangle_\Gamma \arg \left(1 + \frac{g}{f}\right)}^{=0}$$

□

Corollary 1 (Gauss Thm.). *Let $p(z) = a_n z^n + Q(z)$ where $\deg Q \leq n-1$, $a_n \neq 0$, if we take $R >> 1$:*

$$|z| = R \Rightarrow |a_n z^n| > |Q(z)| \Rightarrow N_P\left(\{|z| < R\}\right) N_{a_n z^n}\left(\{|z < R|\}\right) = n$$