Complex Function Theory

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1 Calculating integrals using sums

Let R be a rational function, $\lambda \in \mathbb{R}$. If:

- There are no poles in $\mathbb{R}$.
- $\lambda \neq 0$, $R(x) \leq \frac{C}{1+|x|}$, $x \in \mathbb{R}$.
- $\lambda = 0$, $R(x) \leq \frac{C}{1+|x|^2}$, $x \in \mathbb{R}$.

Then:

$$\int_{-\infty}^{\infty} R(x)e^{i\lambda x}dx = \begin{cases} 2\pi i \sum_{\Im a>0} res_a \left(R(z)e^{i\lambda z}\right) & \lambda \geq 0 \\ 2\pi i \sum_{\Im a<0} res_a \left(R(z)e^{i\lambda z}\right) & \lambda < 0 \end{cases}$$

$$\int_{\Gamma_\rho} R(z)e^{i\lambda z}dz = 2\pi i \sum_{\Im a>0} res_a \left(R(z)e^{i\lambda z}\right)$$

For big enough ρ, where $c_\rho = \left\{z : |z| = \rho, \Im z > 0\right\}$ and $\Gamma_\rho = [-\rho, \rho] + C_\rho$

Lemma 1 (Jordan)**.**

$$\int_{C_\rho} R(z)e^{i\lambda z}dz \xrightarrow{\rho\to\infty} 0$$

Proof.

$$\left|\int_{C_\rho}\right| = \left|\int_0^{\pi} R\left(\rho e^{i\theta}\right) e^{i\lambda\rho e^{i\theta}} \overset{=dz}{i} \rho e^{i\theta} d\theta\right| \leq \int_0^{\pi} \left|R\left(\rho e^{i\theta}\right)\right| e^{-\lambda\rho\sin\theta}\rho d\theta \leq C\int_0^{\pi} e^{-\lambda\rho\sin\theta}d\theta$$

$$= 2c\int_0^{\pi/2} e^{-\lambda\rho\sin\theta}d\theta \leq 2c\int_0^{\pi/2} e^{\frac{2\lambda}{\pi}\rho\theta}d\theta < 2c\overbrace{\int_0^{\infty} e^{\frac{2\lambda}{\pi}\rho\theta}d\theta}^{=\frac{\pi}{2\lambda\rho}\xrightarrow{\rho\to\infty}0} \to 0$$

□

Example 1. Let $\alpha, \beta > 0$, there are poles at $z \pm i\beta$.

$$\int_0^{\infty} \frac{\cos\alpha x}{x^2+\beta^2}dx = \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}+e^{-i\alpha x}}{x^2+\beta^2}dx = \frac{1}{4}\left[2\pi i res_{i\beta}\frac{e^{i\alpha z}}{(z+i\beta)(z-i\beta)} - 2\pi i res_{-i\beta}\frac{e^{-i\alpha z}}{(z+i\beta)(z-i\beta)}\right]$$

$$= \frac{1}{4}\cdot 2\pi i\left[\frac{e^{i\alpha\cdot i\beta}}{i\beta+i\beta} - \frac{e^{-i\alpha(-i\beta)}}{-i\beta-i\beta}\right] = \frac{\pi i}{2}\frac{1}{2i\beta}\left[e^{\alpha\beta}+e^{-\alpha\beta}\right] = \frac{\pi}{2\beta}e^{-\alpha\beta}$$

In particular, if $\beta = 1$ then

$$\frac{2}{\pi}\int_0^{\infty} \frac{\cos\alpha x}{x^2+1}dx = e^{-|\alpha|}$$

Example 2.

$$I=\int_0^\infty \frac{x^n}{1+x^{2n}}dx$$

What is the integral over $\Gamma_R=\{0,R\}\cup\left\{Re^{i\theta}:\theta<{}^{\pi}/{}_{4}\right\}\cup\left\{xe^{i\pi/4}:0\le x\le R\right\}$?

$$\int_{\Gamma_R}\frac{z^n}{1+z^{2n}}dz=2\pi i res_{e^{i\pi/2n}}\left(\frac{z^n}{1+z^{2n}}\right)=2\pi i\frac{e^{i\pi/2}}{2ne^{(2n-1)i\pi/2n}}=-\pi i\frac{i}{ne^{i\pi/2n}}=\frac{\pi}{n}e^{i\pi/2n}$$

As $R\to\infty$, $II=0$, since $|II|\le\frac{C\cdot R}{R^n}\to 0$ and of course,

$$I\to\int_0^\infty\frac{x^n}{1+x^{2n}}dx:=J$$

$$III=-\int_0^R\frac{xe^{i\pi/n}}{1+\left(xe^{i\pi/n}\right)^{2n}}\overset{=dz}{e}^{i\pi/n}dx=e^{i\pi/n}$$

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

Example 3.

$$I(\alpha)=\int_0^\infty\frac{x^\alpha}{1+x}dx,-1<\Re(\alpha)<0$$

Define $\Gamma_{\varepsilon,R}$ as the “key-hole contour”.

$$\int_{\Gamma_{\varepsilon,R}}\frac{z^\alpha}{1+z}dz=2\pi i res_{z=-1}\frac{z^\alpha}{1+z}=2\pi i z^\alpha\Big|_{z=-1=e^{i\pi}}=2\pi i e^{i\pi\alpha}$$

Let $\Gamma_{\varepsilon,R}=-I+II+III+IV$, where

- I is the circle with a radius of ε.
- II is the Line from ε to R just above the real line.
- II is the Line from ε to R just below the real line.
- IV is the circle with a radius of R.

$$II=\int_\varepsilon^R\frac{x^\alpha}{1+x}dx\xrightarrow[\varepsilon\to 0]{R\to\infty}J(\alpha)$$

$$III=-\int_\varepsilon^R\frac{x^\alpha e^{2\pi i\alpha}}{1+x}dx\to e^{2i\pi\alpha}J(\alpha)$$

$$|I|\le 2\pi\varepsilon e^{2\pi|\Im\alpha|}\cdot\frac{\varepsilon^{\Re(\alpha)}}{C}\le C_1e^{1+Re(\alpha)}\xrightarrow{\varepsilon\to 0}0$$

$$|IV|\le 2\pi Re^{2\pi|\Im\alpha|}\frac{R^{\Re(\alpha)}}{R-1}CR^{\Re(\alpha)}\to 0$$

$$\left(1-e^{2\pi i\alpha}\right)J(\alpha)=2\pi i e^{i\pi\alpha}$$

$$J(\alpha)=\frac{e^{i\pi\alpha}}{1-e^{2i\pi\alpha}}2\pi i=-\frac{1}{\frac{e^{\pi i\alpha}-e^{-\pi i\alpha}}{i}}=\frac{\pi}{\sin(\pi\alpha)}$$

Another way of calculating the integral is as follows, letting $x=e^\xi$ and $\zeta=i\nu$:

$$I(\alpha)=\int_0^\infty\frac{x^\alpha}{1+x}dx=\int_{-\infty}^\infty\frac{e^{(\alpha+1)}\xi}{1+e\xi}d\xi=\int_{-\infty}^\infty\frac{e^{\beta\xi}}{1+e^\xi}d\xi$$

Setting Γ_A as the rectangle with vertices at $(-A,0),(A,0),(A,2\pi i)$ and $(-A,2\pi i)$. Then:

$$\int_{\Gamma_A}\frac{e^{\beta\zeta}}{1+e^{\zeta}}d\zeta = 2\pi i res_{\zeta=\pi i}\frac{e^{\beta\zeta}}{1+e^{\zeta}} = 2\pi i\frac{e^{\beta\zeta}}{e^{\zeta}}\Big|_{\zeta=\pi i} = 2\pi i\frac{e^{\beta\pi i}}{-1} - 2\pi i e^{\beta\pi i}$$

$$\int_{\Gamma_A}\int_{-A}^{A}\frac{e^{\beta\zeta}}{1+e^{\zeta}}d\zeta - \int_{-A}^{A}\frac{e^{\beta\zeta}}{1+e^{\zeta}}d\zeta\cdot e^{2\pi i\beta} + I + II$$

Since (H.W) $|I|,|II| \xrightarrow{A\to\infty} 0$ we have $J(\alpha)\left(1-e^{2\pi i\beta}\right) = -2\pi i e^{\pi i\beta}$.

$$\int_{-\infty}^{\infty}\frac{e^{\beta\zeta}}{1+e\zeta}d\zeta = -2\pi i\frac{e^{\pi i\beta}}{1-e^{2\pi i\beta}} = \pi\frac{1}{\frac{e^{\pi i\beta}-e^{-\pi i\beta}}{2i}} = \frac{\pi}{\sin\pi\beta}$$

Corollary 1.

$$\int_{\infty}^{\infty}\text{!!!!!!!!!}$$

1.1 Sums

1.
$$\frac{\pi^2}{\sin^2\pi z} = \sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2}$$

2.
$$\cot\pi z = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

3.
$$\frac{\pi}{\sin\pi z} = \lim_{m\to\infty}\sum_{|n|\le m}\frac{(-1)^n}{z-n}$$

4.
$$\sin\pi z = \pi z\prod_{n\in\mathbb{Z}\setminus\{0\}}\left(1-\frac{z}{n}\right)e^{z/n} = \pi z\prod_{n\ge 1}\left(1-\frac{z^2}{n^2}\right)$$

Proof. 1.
$$\left(\frac{\pi}{\sin\pi z}\right)^2 = \frac{1}{(z-n)^2} + f(z)$$

Where f is an analytic function in a neighborhood of $z=n$. $\left(\frac{\pi}{\sin\pi z}\right)^2$ is 1 periodic, therefore it is sufficient to check that this occurs around $n=0$.
Using the taylor expansion of $\sin\pi z$ we have $\sin\pi z = \pi z + \mathcal{O}\left(z^3\right)$ when $z\to 0$.

$$\frac{1}{\sin\pi z} = \frac{1}{\pi z+\mathcal{O}\left(z^3\right)} = \frac{1}{\pi z} + \mathcal{O}(z),\ z\to 0$$

Meaning that the claim is correct at $n=0$.

$$g(z) := \left(\frac{\pi}{\sin\pi z}\right)^2 - \sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2}$$

g is entire and $g(z+1)=g(z)$, thus g is bounded in $\mathbb{C}$ and as a result of the Liouville theorem g is constant.

$$|\sin\pi z|^2 = |\sin\pi(x+iy)|^2 = \sin^2\pi x + \sinh^2\pi y \xrightarrow{y\to\infty} \infty$$

Thus, as $|y| \to \infty$ we have

$$\left(\frac{\pi}{\sin \pi z}\right)^2 \to 0$$

$$\left|\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}\right| \le \sum_{n\in\mathbb{Z}} \frac{1}{|z-n|^2} \xrightarrow{|y|\to\infty} 0$$

2.

$$(\pi \cot \pi z)' = -\frac{\pi^2}{\sin^2 \pi z}$$

$$\left(\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)\right)' = -\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

Thus,

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) + const$$

Since these two functions are odd, $const = 0$.

3. Leaving this as homework.

4.

$$\frac{(\sin \pi z)'}{\sin z} = \pi \cot \pi z$$

$$\Pi(z) := \pi z \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1 - \frac{z}{n}\right) e^{z/n}$$

Then

$$\frac{\Pi'}{\Pi}(z) = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)$$

Thus, using (2) we have

$$\frac{(\sin \pi z)'}{\sin \pi z} = \frac{\Pi'}{\Pi}(z) \Rightarrow \frac{\sin \pi z}{\Pi(z)} = const$$

$$\sin \pi z = Cz \prod_{n\neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}$$

$$\left.\frac{\sin \pi z}{\Pi(z)}\right|_{z\to 0} \to \pi \Rightarrow C = \pi$$

□