Complex Function Theory

Mikhail Sodin
Arazim ©

December 17, 2015

1 Residues

Let $f \in Hol\left(u_a^*\right)$ and let $g : u_b \xrightarrow[1-1]{onto} u_a$ such that $g(b) = a$, then $f \circ g \in Hol\left(u_b^*\right)$ and $res_a f = res_b\left(f \circ g\right) g'$.

Proof.

$$res_b\left(f \circ g\right) g' = \frac{1}{2\pi i}\int_{|z-b|=\varepsilon} f\left(g\left(z\right)\right) g'\left(z\right) dz = \frac{1}{2\pi i}\int_C f(w)dw = res_a f$$

□

1.1 Residue at ∞

Let $f \in Hol\left(\left\{z : |z| > R\right\}\right)$, then

$$res_\infty f = -\frac{1}{2\pi i}\int_{|z|=R} f(z)dz, \qquad R > \rho$$

Corollary 1. *Let $f \in Hol\left(\mathbb{C}\setminus\left\{a_1, \ldots, a_n\right\}\right)$ then*

$$\sum_{j=1}^{N} res_{a_j} f + res_\infty f = 0 \qquad \max_j\left|a_j\right| < R$$

$$-res_\infty f = \frac{1}{2\pi i}\int_{|z|=R} f(z)dz = \sum_{j=1}^{N} res_{a_j} f$$

Example 1. Given the points $a_1, \ldots, a_N, b_1, \ldots, b_1, \ldots, b_N$ and $R > \max_j\left(\left|a_j\right|, \left|b_j\right|\right)$

$$\frac{1}{2\pi i}\int_{|z|=R} \overbrace{\frac{(z-a_1)\cdots(z-a_N)}{(z-b_1)\cdots(z-b_N)}}^{=f} = \begin{bmatrix} \zeta = \frac{1}{z} \\ dz = -\frac{d\zeta}{\zeta^2} \end{bmatrix} = \frac{1}{2\pi i}\int_{|\zeta|=1/R} \overbrace{\frac{(1-a_1\zeta)\cdots(1-a_N\zeta)}{(1-b_1\zeta)\cdots(1-b_N\zeta)}}^{=g(\zeta)} = \sum_{j=1}^{N}\left(b_j - a_j\right)$$

Where $g \in Hol\left(|\zeta| \leq \frac{1}{R}\right)$

$$g\left(\zeta\right) = 1 + ?\zeta + \ldots$$

$$\frac{g(\zeta)}{\zeta^2} = \frac{1}{\zeta^2} + \frac{?}{\zeta} + \ldots$$

$$? = -\sum_{j=1}^{N} a_j + \sum_{j=1}^{N} b_j$$

Example 2. The residue at infinity of the function $\sqrt{z^2-1} \in Hol\left(\mathbb{C}\setminus[-1,1]\right)$

$$\frac{1}{2\pi i}\int_{|z|=2}\sqrt{z^2-1}dz = -res_\infty\sqrt{z^2-1} = -\frac{1}{2}$$

This is because

$$\sqrt{z^2-1} = z\sqrt{1-\frac{1}{z}} = z\left(1-\frac{1}{2z}+\dots\right) = z-\frac{1}{2z}+\dots$$

1.2 Integrals

Example 3. Let R be a rational function.

$$\int_{-\pi}^{\pi} R\left(\cos\theta,\sin\theta\right)d\theta = \frac{1}{i}\int_{|z|=1} R\left(\frac{z+z^{-1}}{2},\frac{z-z^{-1}}{2i}\right)\frac{dz}{z}$$

Example 4. Let $a\in\mathbb{R}$ and $|a|>1$

$$\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} = \frac{1}{i}\int_{|z|=1}\frac{dz}{z\left(a+\frac{z+z^{-1}}{2}\right)} = \frac{2}{i}\int_{|z|=1}\frac{dz}{1+2az+z^2} \stackrel{\star}{=}$$

$$z_{1,2} = -a\pm\sqrt{a^2-1}, z_1\cdot z_2 = 1, |\overbrace{-a+\sqrt{a^2-1}}^{=z_1}| < 1 \qquad \left|-a-\sqrt{a^2-1}\right| = \frac{1}{\left|-1+\sqrt{a^2-1}\right|} > 1$$

$$\stackrel{\star}{=} 4\pi\frac{1}{2\pi i}\int_{|z|=1}\frac{dz}{\left(z-z_1\right)\left(z-z_2\right)} = 4\pi res_{z=z_1}\frac{1}{\left(z-z_1\right)\left(z-z_2\right)} = 4\pi\frac{1}{z_1-z_2} = \frac{4\pi}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$$

Example 5 (Fourier)**.**

$$\int_{-\infty}^{\infty} R\left(x\right)e^{ix\lambda}dx = \begin{cases} 2\pi i\int_{a:\Im a>0} res_a\left[R(z)e^{i\lambda z}\right] & \lambda\geq 0\\ -2\pi i\int_{a:\Im a<0} res_a\left[R(z)e^{i\lambda z}\right] & \lambda\leq 0\end{cases}$$

R is rational such that there are no poles in $\mathbb{R}$ and $\Im\lambda = 0$. If $\lambda\neq 0$ then $R\left(z\right) = \mathcal{O}\left(\frac{1}{z}\right), z\to\infty$ and if $\lambda = 0$ then $R\left(z\right) = \mathcal{O}\left(\frac{1}{z^2}\right), z\to\infty$. Let $C_\rho = \left\{|z| = \rho, \Im z\geq 0\right\}$

$$\int_\rho^\rho R(x)e^{1\lambda x}dx + \int_{C_\rho} R\left(z\right)e^{i\lambda z}dz = \sum_{\substack{a:|a|<\rho\\ \Im a>0}} res_a\left[R(z)e^{i\lambda z}\right]$$

Where the last inequality comes from Cauchy's theorem.

$$\lim_{\rho\to\infty}\int_{-\rho}^{\rho} R\left(x\right)e^{i\lambda x} = -\lim i_{\rho\to\infty}\int_{C_\rho} R(z)e^{i\lambda z}dz + 2\pi i\cdot\sum_{a:\Im a>0} res_a\left(R(z)e^{i\lambda z}\right)`$$