Complex Function Theory

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1 Isolated singularities

Continuing from last lesson

Definition 1. Let $f \in Hol\left(U_a^*\right)$ then A is an isolated singularity.[1]

1. a is called a removable singularity id there exists a $F \in Hol\left(u_a\right)$ and $F = f$ in u_a^*
2. a is a pole of f if $\lim_{z \to a} \left| f(z) \right| = \infty$

Theorem 1 (Removable singularity)**.** *Let $f \in Hol\left(u_a^*\right)$ and is bounded there. ($\exists M \sup_{u_a^*} |f| < M$) then a is a removable singularity.*

Proof. Using the Taylor expansion for f , $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ where$a_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(Z)}{(z-a)^{n+1}} dz$. We want to check that for all $n \in \mathbb{N}$, $a_{-n} \overset{??}{=} 0$.

$$|a_{-n}| = \left| \frac{1}{2\pi i} \int_{|z-a|=\rho} f(z)(z-a)^{n-1} dz \right| \le M \frac{1}{2\pi} \rho^{n-1} \cdot 2\pi\rho = M\rho^n]$$

Since $\rho \downarrow 0$ we have that the equality holds and $f(z) = \sum_{n \ge 0} a_n \left(z-a\right)^n$ □

Note 1. We will define a $M_f\left(\rho\right) = \max_{|z-a|=\rho} \left| f(z) \right|$. We will assume that $\lim_{\rho \downarrow 0} \rho^{M_f}\left(\rho\right) = 0$, then a is a isolated singularity.

1.1 Poles

a is a pole of f if $\lim_{z \to a} \left| f(z) \right| = \infty$ and if we define a function $g := \frac{1}{f}$ then $g \in Hol(u_a^*), \left| g(z) \right| \xrightarrow{z \to a} 0 \Rightarrow g \in Hol(u_a)$

- a is an isolated singularity of g by **??**.
- $g(a) = 0 \Rightarrow g(z) = h(z)(z-a)^m, h \in Hol(u_a)$ and $h(a) \neq 0$, where m is the multiplicity of m of g, $h(a) = \frac{g^{(m)}(a)}{m!} \Rightarrow f(Z) = \frac{r(z)}{(z-a)^m}, q = \frac{1}{n}, q \in Hol(u_a), q(a) \neq 0$

a is a pole $\Rightarrow f(z) = \frac{q(z)}{(z-a)^m}$, m is the multiplicity of f at point $a \Rightarrow f(z) = \sum_{n \ge -m} a_n \left(z-a\right)^n$

[1]Using the markings from Calculus 1 where we denote u_a as an area of a and $U_a^* = U_a \setminus \{a\}$ is a punctured area of a.

1.2 Essential singularities

An essential singularity is neither a pole or an isolated singularity.

Theorem 2 (Sokhotski-Casoratti-Wejerstrass). *If a is an essential singularity then for every punctured neighborhood u_a^* such that $f \in Hol\,(u_a^*)$, $f(u_a^*) = \mathbb{C}$*

Proof. WE will assume that $dist(f(u_a^*), w) \geq \delta \left(\Leftrightarrow w \notin f(u_a^*)\right.$, $g(z) := \frac{1}{f(z)-w}$.
Since $g\ inHol(u_a^*)$ and $|g| \leq \frac{1}{\delta}$ in u_a^*, by the first theorem we have that a is an isolated singularity of $g(g \in Hol(u_a))$.
Now, since $f(z) = w + \frac{1}{g(z)}$ then

- $g(a) = 0 \Rightarrow a$ is a pole of f.
- $g(a) \neq 0 \Rightarrow a$ is an isolated singularity of f

And thus, we have arrived at a contradiction to the fact that a is an essential singularity. □

Now we will look at punctured neighborhood of ∞, $\{|z| > R\}$ (R is large). Then, if $F \in Hol\left(\{|z| > R\}\right) \Leftrightarrow f(\zeta) = F(\frac{1}{\zeta})$, $f \in Hol\left(\{0|\zeta| < R\}\right)$.

Definition 2. F is analytic in ∞ if f is analytic at $\zeta = 0$ and F has a pole at ∞ if f has a pole at 0.

Corollary 1. *1. $f \in Hol\left(\hat{\mathbb{C}}\right) \Rightarrow F = const$ (F is analytic at $\infty \Rightarrow$ bounded in $C \overset{Liouville}{\Rightarrow} f = const$)*

2. Let $F \in Hol(\mathbb{C})$ and has a pole at $\infty \Leftrightarrow F$ is a polynomial.

Proof. In the first direction, let F be a polynomial,

$$F(z) = f(\frac{1}{z}) = \sum_{n \geq -m} a_n \left(\frac{1}{z}\right)^n = \sum_{n \leq m} c_n z^n, c_n = a{-}n$$

In the other direction, if F is analytic at 0 then $c_n = 0$ for all $n < 0$ leading us to $f(z) = \sum_{0 \leq n \leq m} c_n z^n$ □

3. Let $F \in Hol\left(\hat{C} \setminus \{a_1, a_2, \ldots, a_N\}\right)$ where $a_1, \ldots, a_N$ are poles of F. Then F is a rational function $F = \frac{P}{Q}$ and P, Q are polynomials. We will look at $f_1(z) - F(z)(z-a_1)^{m_1} \ldots (z-a_{N-1})^{m_{N-1}}$ where m_j is the order of the pole at the point a_j.

- *$f_1 \in Hol(\mathbb{C}) \Rightarrow F_1 = P$*
- *δ, f_1 has a pole at ∞ $F(z) = \frac{P(z)}{\prod_{j=1}^{N-1}(z-a_j)^{m_j}}$ $0 \to \infty$.*

2 Residues

Let $f \in Hol\,(U_a^*)$ and $\bar{\mathcal{D}}\,(a, \varepsilon) \subset u_a$.

$$res_a f := \frac{1}{2\pi i} \int_{|z-a|=\varepsilon} f(z)dz = c_{-1}$$

Theorem 3 (Cauchy)**.** *Let $G \subset \mathbb{C}$ be a good domain, $a_1, \dots, a_N \in G$ and let $f \in Hol\left(G \backslash \{a_1, \dots, a_N\}\right), f \in C\left(\bar{G} \backslash \{a_1, \dots, a_N\}\right)$ then*

$$\int_{\partial G} f(Z)dz = 2\pi i \sum_{j=1}^{N} res_{a_j} f$$

Proof. Taking $\varepsilon < \min_j dist\left(a_j, \partial G\right)$ and $\varepsilon < \min_{i \neq j}\left(\left|a_i - a_j\right|\right)$, then we will define

$$G_\varepsilon = G \backslash \bigcup_{j=1}^{N} \bar{\mathcal{D}}\left(a_j, \varepsilon\right)$$

$$f \in Hol\left(G_\varepsilon\right) \cap C\left(\bar{G}_\varepsilon\right) \Rightarrow \int_{\partial G_\varepsilon} f(z)dz = 0 = \int_{\partial G} f dz - \sum_{j=1}^{N} \int_{\partial \mathcal{D}\left(a_j, \varepsilon\right)} f dz$$

□

The last value is equal to the residue at each point a_j, meaning that these are important, we will look at how to find them.

Corollary 2. *1. If $f = \frac{\varphi}{\psi}, \varphi, \psi \in Hol\left(u_a\right)$ and ψ has a simple zero at $z = a$ (multiplicity = 1) $\Rightarrow$*

$$red_a f = \frac{\varphi(z)}{\psi'(a)}$$

Proof.

$$c_{-1}(f) = \lim_{z \to a}\left(z - a\right) f(z) = \lim_{z \to a} \frac{\varphi(z)}{\frac{\psi(z) - \psi(a)}{(z-a)}} = \frac{\varphi}{\psi'(a)}$$

Where the second equality is because $\varphi(a) = 0$. □

2. $f(z) = \frac{\varphi(z)}{(z-a)^m}$, $\varphi \in Hol\left(u_a\right)$, then

$$c_{-1}\left(f\right) = \frac{\varphi^{(m-1}(a)}{(m-1)!}$$

Proof.

$$\varphi(z) = \sum_{n \geq 0} \frac{\varphi^{(n)}(a)}{n!}(z - a)^n$$

$$f(z) = \sum_{n \geq 0} \frac{\varphi^{(n)}(a)}{n!}(z - a)^{n-m}$$

□

Example 1. 1.

$$f(z) = \frac{z}{(z-1)(z-2)^2}$$

$$res_1 f = \frac{z}{(z-2)^2}\Big|_{z=1} = \frac{1}{1} = 1$$

$$res_2 f = \left(\frac{z}{z-1}\right)'_{z=2} = \left(1 + \frac{1}{z-1}\right)'_{z=2} = -\frac{1}{(z-1)^2}\Big|_{z=2} = -1$$

2.

$$f(z)=\frac{g(z)}{1+z^n}, g\in Hol(\mathbb{C}), \omega^n=-1$$

$$res_\omega f=\frac{g(\omega)}{n\omega^{n-1}}=-\frac{1}{n}\omega g(\omega)$$

$$\omega^{n-1}=\frac{1}{\omega}\omega^n=-\frac{1}{\omega}$$

3. $g\in Hol(G), g(a)=0$ and $f=\frac{g'}{g}$.

$$g(a)=0,\ g(z)=(z-a)^m\,h(z),\ h\in Hol\,(G)\,,\ h(a)\neq 0$$

$\frac{h'}{h}$ is analytic at a, therefore

$$f(z)=\frac{m}{z-a}+\frac{h'}{h}\Rightarrow res_a\frac{g'}{g}=-m$$

Note 2.

$$g(z)=\frac{h(z)}{(z-a)^m}\Rightarrow res_a\frac{g'}{g}=-m$$

4. $f(z)=\tan z,\ z_k=\frac{\pi}{2}=k\pi, k\in\mathbb{Z}$. Then

$$res_{z_k}\tan z=\frac{\sin z_k}{-\sin z_k}=-1=\frac{\sin z}{\cos z}$$

$$\tan z=\sum_{n\geq 0}c_nz^n \qquad a_n=c_n\left(\frac{\pi}{2}\right)^n$$

We will define the odd function f as follows:

$$f=\tan\frac{\pi z}{2}=\sum_{n\geq 0}a_nz^n=\sum_{n\geq 0}a_{2n+1}z^{2n+1}$$

Setting $z=\pm 1$, $res_{\pm 1}f=-1$

$$g(z)=f(z)-(-1)\left(\frac{1}{z-1}+\frac{1}{z+1}\right)=f(z)-\frac{2z}{1-z^2}$$

$$\tan\frac{\pi z}{2}=\frac{\sin\frac{\pi z}{2}}{\cos\frac{\pi z}{2}}=-\frac{2}{\pi}$$

And of course,

$$\tan\frac{\pi z}{2}=\frac{4}{\pi}+\frac{z}{1-z}+g(z)$$

Since $g\in Hol\left(\{|z|<\varepsilon\}\right)$ $g(z)=\sum_{n\geq 0}d_{2n+1}z^{2n+1}$ and $a_{2n+1}=\frac{4}{\pi}+d_{2n+1}=\frac{4}{\pi}+o(1), n\to\infty$

$$\frac{1}{\varepsilon}=\overline{\lim_{n\to\infty}}|d_{2n+1}|^{1/2n+1}$$

$d_{2n+1}\to 0$, $\forall A<\varepsilon, |d_{2n+1}|\,CA^{-2n}$

If we have a function f that is analytic in a punctured neighborhood of a, and $g \in Hol(u_b)$, $g : u_b \xrightarrow[onto]{1-1} u_a$, where $g(z) = a$. In addition we will define a function $h(z) = (f \circ g)(z)$ and $h \in Hol\left(u_b^*\right)$. If we look at the residue of h at the point b

$$res_b h = \frac{1}{2\pi i} \int_{|w-b|=\varepsilon} h(w)dw \neq res_a f = \frac{1}{2\pi i} \int_{|z-a|=\rho} f(z)dz$$

However if we change the variable as follows:

$$\int_{|w-b|=\varepsilon} f(g(w))g'(w)dw = \int_1^{2\pi 1} \int_{g\left(\{|w-b|=\varepsilon\}\right)} f(z)dz = \frac{1}{2\pi i} \int_{|z|=\rho} f(z) = res_a f$$

Then we get

$$res_a f = res_b (f \circ g) g'$$