Complex Function Theory

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1 Laurent expansion

1.
$$a \in \mathbb{C} \qquad f\left(z\right)=\frac{1}{z-a}$$

- $|z|>|a|$:
$$f(z)=\frac{1}{z}\frac{1}{1-\frac{a}{z}}=\sum_{n\geq 0}\frac{a^n}{z^{n+1}}$$

- $|z|<|a|$:
$$f(z)=-\frac{1}{a}\frac{1}{1-\frac{a}{z}}=-\sum_{n\geq 0}\frac{z^n}{a^{n+1}}$$

2.
$$f(z)=\frac{1}{(z-2)(z-1)}=\frac{1}{z-2}-\frac{1}{z-1}$$

Here we have threee different domains:

- $|z|<1$
$$f(z)=-\sum_{n\geq 0}\frac{z^n}{2^{n+1}}+\sum_{n\geq 0}z^n=\sum_{n\geq 0}\left(1-\frac{1}{2^{n+1}}\right)z^n$$

- $1<|z|<2$
$$f(z)=-\sum_{n\geq 0}\frac{z^n}{2^{n+1)}}-\sum_{n\geq 0}\frac{1}{z^{n+1}}=\sum_{n\in\mathbb{Z}}c_nz^n \qquad c_n=\begin{cases}-\frac{1}{2^{n+1}} & n\geq 0\\ -1 & n<0\end{cases}$$

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$$f(z)=\sum_{n\geq 0}\frac{2^n}{z^{n+1}}-\sum_{n\geq 0}\frac{1}{z^{n+1}}=\sum_{n=-\infty}^{-1}\left(2^{1+n}-1\right)z^n=\sum_{n\leq -2}\left(2^{1+n}-1\right)z^n$$

Definition 1 (Laurent series)**.**

$$\sum(z):=\sum_{n\in\mathbb{Z}}c_nz^n=\sum_{n\geq 0}{}_{+}+\sum_{n\leq -1}{}_{-} \qquad \left(\sum_{n\in\mathbb{Z}}c_n\left(z-z_0\right)^n\right)$$

$$\sum{}_{+} \qquad |z|<R_+=\left(\overline{\lim_{n\to\infty}}|c_n|^{1/n}\right)$$

$$\sum\nolimits_{-} \qquad |z| > R_- = \left(\overline{\lim_{n\to\infty}}|c_{-n}|^{1/n}\right)$$

$R_+ > R_-$, $R_- < \rho < R_+$

$$c_n = \frac{1}{2\pi i}\int_{|z|=\rho}\frac{\sum(z)}{z^{n+1}}dz \tag{1}$$

$\sum$ converges uniformly in $\{|z| = \rho\}$

Theorem 1 (Laurent theorem)**.** *If $f \in Hol(R_- < |z| < R_+)$ then*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n z^n$$

Converges uniformly and locally and **??** *holds.*

Proof. Let $R_- < \rho_- < \rho_+ < R_+$, $G = \{\rho_- < |z| < \rho_+\}$ nad $f \in Hol(G) \cap C(\bar{G})$, $z \in G \Rightarrow$

$$f(z) = \frac{1}{2\pi i}\int_{\partial G}\frac{f(\zeta)}{\zeta - z}d\zeta = \overbrace{\frac{1}{2\pi i}\int_{|\zeta|=\rho}\frac{f(\zeta)}{\zeta - z}d\zeta}^{f_+} - \overbrace{\frac{1}{2\pi i}\int_{|\zeta|=\rho_-}\frac{f(\zeta)}{\zeta - z}d\zeta}^{f_-}$$

$f_+ \in Hol(|z| < \rho_+)$, $f_- \in Hol(|z| > \rho_-)$.

$$f_+(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho_+}\frac{f(\zeta)}{\zeta - z}d\zeta \overset{\star}{=} \sum_{n\ge 0}\left(\frac{1}{2\pi i}\int_{|\zeta|=\rho}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right)z^n$$

$$f_-(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho_-}\frac{f(\zeta)}{\zeta - z}d\zeta \overset{\star\star}{=} \sum_{n\ge 0}\left(\frac{1}{2\pi i}\int_{|\zeta|=\rho}f(\zeta)\zeta^n d\zeta\right)z^n = -\sum_{n\le -1}\left(\frac{1}{2\pi i}\int_{|\zeta|=\rho_-}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right)$$

$$(\star) \qquad \frac{1}{z-\zeta}$$

$$c_n = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{f\left(e^{i\theta}\right)}{e^{i(n+1)\theta}}!!!!$$

□

2 Isolated singularities

Definition 2. Let $f \in Hol(U_a^*)$ then A is an isolated singularity.[1]

1. a is called a removable singularity id there exists a $F \in Hol(u_a)$ and $F = f$ in u_a^*
2. a is a pole of f if $\lim_{z\to a}|f(z)| = \infty$

[1] Using the markings from Calculus 1 where we denote u_a as an area of a and $U_a^* = U_a \setminus \{a\}$ is a punctured area of a.