Complex Function Theory

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1 Corollaries from the Cauchy equation

Reminder 1. If $G \subset \mathbb{C}$ is a good domain, $\Gamma = \partial G, z \in G$ and $g \in Hol\left(G\right) \cap C\left(G\bar{G}\right)$ then

$$f\left(z\right) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f\left(\zeta\right)}{\zeta - z}d\zeta$$

Lemma 1. *Let $\gamma : I \to \mathbb{C}$ be piecewise C^1, regular and $\varphi \in C\left(\gamma\right)$. We will define*

$$F\left(z\right) := \int_{\gamma}\frac{y\left(\zeta\right)}{\zeta - z}d\zeta$$

Then $F \in Hol\left(\mathbb{C}\backslash\gamma\right), F \in C_{\mathbb{C}}^{\infty}\left(\mathbb{C}\backslash\gamma\right)$ $z \in \mathbb{C}\backslash\gamma$, then:

$$F^{(n)}(z) = n!\int_{\gamma}\frac{\varphi(\zeta)d\zeta}{\left(\zeta - z\right)^{n+1}}$$

Proof. We will set $z_0 \in \mathbb{C}\backslash\gamma$, $\delta := dist\left(z_0, \gamma\right)$ meaning that $\delta > \left|z - z_0\right| \left(\Rightarrow z \in \mathbb{C}\backslash\gamma\right)$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\cdot\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n\geq 0}\frac{\left(z - z_0\right)^n}{\left(\zeta - z_0\right)^{n+1}}$$

Using the Majorant theorem from Calculus 2 we can prove that the series converges uniformly, and indeed:

$$\zeta \in \gamma \qquad \left|\frac{z - z_0}{\zeta - z_0}\right| < 0$$

Thus:

$$\frac{\varphi(\zeta)}{\zeta - z} = \sum_{n\geq 0}\left(z - z_0\right)^n\frac{\varphi(\zeta)}{(\zeta - z_0)^{n+1}}$$

Setting

$$F\left(z\right) = \sum_{n\geq 0}\left(z - z_0\right)^n\overbrace{\int_{\gamma}\frac{\varphi\left(z_0\right)}{\left(\zeta - z_0\right)^{n+1}}}^{:=c_n}$$

$F \in Hol\left(\left\{z : \left|z - z_0\right| < \delta\right\}\right)$

$$\frac{F^{(n)}(z_0)}{n!} = \int_{\gamma}\frac{\varphi(\zeta)d\zeta}{\left(\zeta - z_0\right)^{n+1}}$$

□

Corollary 1. *1.* $f \in C^{\infty}_{\mathbb{C}}(G)$ *and*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|\zeta - z| = \rho} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Where $\rho < dist\,(z, \partial G)$

2.

$$f(w) = \sum_{n \geq 0} \frac{f^{(n)}(z)}{n!}(w - z)^n \qquad |w - z| < dist\,(z, \partial G)$$

Corollary 2 (Morera theorem)**.** *Let* $f \in C(G)$ *such that for every closed triangle* $T \subset G$ *for which* $\int_{\partial T} f dz = 0$ *then* $f \in Hol(G)$

Proof. There exists a previous function $F \in Hol(G)$ such that $F' = f$. From the first corollary, $f \in Hol(G)$. □

Corollary 3. *Let* $G \subset C$ *be a domain and* $I \subset G$ *is a closed interval. If* $f \in Hol\left(G \backslash I\right)$ *and is continuous in* G *then* $f \in Hol\,(G)$

Corollary 4. *Defining* $\mathcal{D} = \left\{z : |z - a| < \rho\right\}$ *we have the* ***Cauchy inequalities:***

$$F \in Hol\left(\mathcal{D}\,(a, \rho)\right) \cap C\left(\hat{\mathcal{D}}\,(a, \rho)\right) \Rightarrow \left|f^{(n)}(a)\right| \leq \frac{n!}{\rho^n} \cdot \max_{\partial \mathcal{D}(a, \rho)} |f|$$

Proof.

$$\left|f^{(n)}(a)\right| = \left|\frac{n!}{2\pi i}\int_{\partial \mathcal{D}(a,\rho)} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta\right| \leq \frac{n!}{2\pi} \max_{\zeta \in \partial \mathcal{D}(a, \rho)} \frac{\left|f(\zeta)\right|}{\left|\zeta - a\right|^{n+1}} \cdot \pi\rho$$

□

Corollary 5 (Liouville theorem)**.** *Let* $f \in Hol\,(\mathbb{C})$ *and bounded* $\Rightarrow f \equiv const$

Proof. $z \in \mathbb{C}$

$$\left|f'(z)\right| \leq \frac{1}{\rho} \overbrace{\max_{\partial \mathcal{D}(z, \rho)} |f|}^{\leq C = \sup_{\mathbb{C}}|f|} \Rightarrow f'(z) = 0 \Rightarrow f \equiv const$$

□

Corollary 6 (Fundamental theorem of algebra)**.** *Let* $p \in \mathbb{C}[z]$ *be a polynomial,* $\deg P \geq 1$ *then there exists a point* $z_0 \in \mathbb{C}$ *such that* $p(z_0) = 0$

Proof. WE will assume by contradiction that $p \neq 0$ in $\mathbb{C} \Rightarrow \frac{1}{p} \in Hol\,(\mathbb{C})$.

$$\left|p(z)\right| \xrightarrow{z \to \infty} \infty \Rightarrow \left|\frac{1}{p(z)}\right| \xrightarrow{z \to \infty} = 0$$

Therefore $\frac{1}{p}$ is bounded in $\mathbb{C}$ and as a result is constant, $p \equiv const \Rightarrow degp = 0$. □

2 Uniqueness theorems

Theorem 1. *Let $f \in Hol(G)$ and $\exists a \in G$ such that $f^{(m)}(a) = 0, m \geq 0$ then $f \equiv 0$.*

Proof.

$$Z = \left\{ z \in G : \forall m \geq 0 \, f^{(m)}(z) = 0 \right\}$$

1. $z \neq \emptyset$.
2. Z is open. $f|_{\mathcal{D}(z,\delta)} \equiv 0$(Taylor expansion around z).
3. $G \backslash Z$ is open (continuity of $f^{(m)}$).

$z \in G\backslash Z \Rightarrow \exists m \geq 0 : f^{(m)}(z) \neq 0 \Rightarrow f^{(m)} \neq 0$ in a neighborhood of z. By connectivity of G, he have $Z = G$ □

Claim 1. $f(z) = f'(z) = \cdots = f^{(m-1}(a) = 0 \Rightarrow f(Z) = (z-a)^m g(z)$, where

$$g(a) = \frac{f^{(m)}(a)}{m!}, \, g \in Hol(\text{Neighborhood of a})$$

Proof.

$$f(z) = \sum_{n \geq m} \frac{f^{(n)}(a)}{n!}(z-a)^n = (z-a)^m \overbrace{\sum_{n \geq 0} \frac{f^{(n+m)}(a)}{(n+m)!}(z-a)^n}^{:=g(z)}$$

□

Definition 1 (multiplicity)**.** The multiplicity of zero at a point a is $n \in \mathbb{Z}_+$ such that $f(a) = \cdots = f^{(n-1)}(a) = 0$ $f^{(n)} \neq 0$ e.g. $m_f(a) = \min \left\{ n \in \mathbb{Z}_+ : f^{(n)}(a) \neq 0 \right\}$.

Note 1. $f(a) \neq 0 \Rightarrow m_f(a) = 0$, $m_f(a) = \infty \Rightarrow f \equiv 0$

Theorem 2. *Let $\{z_n\} \subset G$ converges to $a \in G$. We will assume that $f(z_n) = 0$ then $f \equiv 0$.*

Proof. '?$f^{(m)}(a) = 0$? We will prove with induction on m. For $m = 0$ it is obvious (continuity of f).

Induction step: $m-1 \Rightarrow m$ $f(a) = f'(a) = \cdots = f^{(m-1)}(a) \Rightarrow g(z) = \frac{f(z)}{(z-a)^m}$ is analytic in G. $f(z_n) = 0 \Rightarrow g(z_n) = 0$, thus by continuity of g we have $g(a) = 0 \Rightarrow f^{(m)}(a) = 0$. □

Corollary 7. *If $f_1(z_n) = f_2(z_n)$ for some $\{z_n\} \subset G$ then $f_1 = f_2$ in G.*

Theorem 3 (Weierstrauss)**.** *If $\{f_n\} \subset Hol(G)$ uniformly converges locally then* $\lim_n f_n = f \in Hol(G)$ *and for all $j \geq 0$ $f_n^{(j)} \xrightarrow{u} f^{(j)}$ locally.*

Proof. Setting K, $\delta = \frac{1}{2} dist\,(K, \partial G) > 0$ $K_{+\delta} = \left\{ z : dist(z, K) \leq \delta \right\} \subset G$ is compact.

1. $f \in Hol(G)$: $T \subset G$ is a closed triangle, $\partial T \subset G$ is compact.

$$\int_{\partial T} f_n dz = 0 \Rightarrow \int_{\partial T} f dz = 0 \Rightarrow f \text{ is analytic in } G$$

2. $f^{(j)} \xrightarrow{u} f^{(j)}$ in K.
   Let $z \in K$

$$\left|f_n^{(j)}(z) - f^{(j)}(z)\right| \le \frac{j!}{\delta^j} \max_{|\zeta - z| = \delta} \left|f_n(\zeta) - f(\zeta)\right| \overset{\text{Cauchy}}{\le} \frac{j!}{\delta^j} \max_{K_{+\delta}} |f_n - f| \xrightarrow{n\to\infty} 0$$

$$\max_K \left|f_n^{(j)} - f^{(j)}\right| \le \frac{j!}{\delta^j} \max_{K_{+\delta}} |f_n - f| \Rightarrow f_n^{(j)} \xrightarrow{u} f^{(j)} \text{ (in } K)$$

□

An example of a function that is not zero in C^∞ but the multiplicity of 0 at 0 is ∞ in $\mathbb{R}$:

$$f(x) = \begin{cases} e^{\frac{1}{x^2}} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

Problem 1. $I = [a,b] \subset \mathbb{R}$ $f : I \to \mathbb{C}$ such that for all $n \ge 0$, $\max_I \left|f^{(n)}\right| \le C^n n$ and $C > 0$ then there exists a domain $I \subset G$ such that $f \in Hol\,(G)$.

Problem 2 (S. bernstein)**.** $f : [0,1) \to \mathbb{R}$, $f \in C^\infty(I)$ and assume that $f^{(n)}(x) \ge 0$ for all $x \in [0,1), n \in \mathbb{Z}_+$ then

$$f \in Hol\left(\left\{|z| < 1\right\}\right)$$