Complex Function Theory

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Continuing from last lesson, we need to prove that

$$C_R^+ = \left\{z = Re^{i\theta}, 0 \le \theta \le \pi\right\} = \int_{C_R^+} \frac{e^{iz}}{z} dz \xrightarrow[R\to\infty]{???} 0$$

In order to prove that, we will use the fact that $\left|e^A\right| = e^{\Re A}, A = iRe^{i\theta}$ and $\Re A = -R\sin\theta$. Thus :

$$= \int_\pi^0 \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} \overbrace{Re^{i\theta}}^{\frac{dz}{d\theta}} d\theta \Rightarrow \left|\int_{C_R^+}\right| \le \int_0^\pi e^{-R\sin\theta} d\theta = s\int_0^{\pi/2} e^{-Rs\in\theta} d\theta \le 2\int_0^{\pi/2} e^{-\frac{2}{\pi}R\theta} d\theta < 2\int_0^\infty e^{-\frac{2}{\pi}R\theta} d\theta = \frac{2\pi}{2R} \xrightarrow{R\to\infty} 0$$

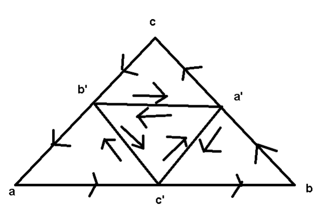
Where the second to last inequality comes from the fact that $\sin\theta \ge \frac{2}{\pi}\theta, 0 \le \theta \le \frac{\pi}{2}$.

1 "Cauchy theorem"

Theorem 1 (E. Goursat)**.** *Let $G \subset \mathbb{C}$ be a domain, $f \in Hol(G)$ and $T \subset G$ is a closed triangle, then*

$$\int_{\partial T} f dz = 0$$

Proof. Let $I = \int_{\partial T} f dz \ne 0$. We will divide the triangle into four similar triangles by splitting it at the medians.



Then we have

$$\int_{\partial T} f dz = \int_{k=1}^4 \int_{\partial T_k} f dz$$

And there exists a k such that

$$\left|\int_{\partial T_k} f dz\right| \ge \frac{1}{4}|I|$$

Splitting these triangles repeatedly we have $T^{(0)} = T$ and $T^{(1)} = T_k$. As a result we have a series of nested compact sets $T^{(0)} \supset T^{(1)} \supset \cdots \supset T^{(j)} \supset \dots$ and of course,

$$\left|\int_{\partial T^{(j)}} f dz\right| \geq \frac{1}{4^j}|I| \qquad L\left(\partial T^{(j)}\right)\frac{1}{2^j}L\left(\partial T\right)$$

As a result of Cantors lemma we have

$$\bigcap_{j\geq 0} T^{(j)} = \{z_0\}$$

$$f\left(z\right) = f(z_0) + f'\left(z_0\right)\left(z - z_0\right) + o\left(z - z_0\right), z \to z_0$$

Let $\varepsilon > 0$. For $j \geq j_0\left(\varepsilon\right)$ we will take $z \in T^{(j)}$

$$\left|f(z) - f(z_0) - f'\left(z_0\right)\left(z - z_0\right)\right| < \varepsilon|z - z_0| \leq \varepsilon\text{diam}\left(T^{(j)}\right)$$

$$\left|\int_{\partial T^{(j)}} f(z)dz\right| - \left|\int_{\partial T^{(j)}} f(z) - f(z_0) - f'(z_0)\left(z - z_0\right)dz\right| = L\left(\partial T^{(j)}\right)\cdot\varepsilon\text{diam}\left(T^{(j)}\right) = \varepsilon L\left(\partial T^{(j)}\right)^2 = \varepsilon\frac{L\left(\partial T\right)}{4^j}$$

We have gotten

$$\frac{1}{4^j}|I| \leq \left|\int_{\partial T^{(j)}} f dz\right| \leq \varepsilon \cdot \frac{L^2\left(\partial T\right)}{4^j} \Rightarrow |I| < \varepsilon L^2\left(\partial T\right) \Rightarrow I = 0$$

□

Definition 1. G is a convex domain if

$$\forall z_1, z_2 \in G \qquad [z_1, z_2] \subset G \qquad \left\{z = (1 - t)\,z + tz_2 : 0 \leq t \leq 1\right\}$$

Theorem 2. *Let G be a convex domain and $f \in Hol\left(G\right), \gamma : [0, 1] \to G$ is a closed arc, regular and a piecewise C^1 function then*

$$\int_\gamma f dz = 0$$

Claim 1. There exists a previous function $F \in Hol(G)$ such that $F' = f$

proof of claim. Let $z_0, z_1 \in G$, we will define

$$F\left(z\right)\int_{[z_0.z]} f(\zeta)d\zeta,\ z \in G$$

Does $F' = f$?? By 1 we have

$$F(z) - F(z_1) = -\int_{[z,z_1} f(\zeta)d\zeta$$

Taking a z_1 such that for $\varepsilon > 0$. $\left|f(\zeta) - f(z)\right| < \varepsilon$.

$$\left|\frac{F(z) - F(z_1)}{z - z_1} - f(z)\right| = \left|\frac{1}{z - z_1}\int_{[z_1,z]} f(\zeta)d\zeta - f(z)\right| = \left|\frac{1}{z - z_1}\int_{[z_1,z]}\left[f(\zeta) - f(z)\right]d\zeta\right| \leq \frac{1}{|z - z_1|}\cdot\varepsilon|z - z_1| = \varepsilon$$

□

Definition 2 ("good domain")**.** G is a good domain if its closure can be expressed as

$$\partial G = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_p$$

Where all Γ_j are disjoint,regular and piecewise C^1.

Theorem 3. *Let G be a good domain, $f \in Hol(G) \cap C\left(\bar{G}\right)$ then*

$$\int_{\partial G} f dz = 0$$

Proof. We will split the plane into a grid where each open square is marked with a Q_j, then using a claim not formally proved yet we have

- $$\# \left\{ \partial G \bigcap \left(\bigcup_j \partial Q_j \right) \right\} < \infty$$
- Q_j is a inner square if $\bar{Q}_j \subset \bar{G}$.
- Q_j is a boundary square if $Q_j \cap \partial G \neq \emptyset$.
- We will define $G_j = Q_j \cap G$

$$\int_{\partial G_2} f dz = \sum \int_{\partial G_j} f dz \text{ (continuity of } f)$$

Using 1 we have that if Q_j is an inner square then

$$\int_{\partial Q_j} f dz = 0$$

As a result the previous sum is equal to the following

$$\sum_{j:Q_j \text{inner}} \int_{\partial G_j} f dz = \sum_{j:Q_j \text{inner}} \int_{\partial G_j} \left[f(z) - f(z_j) \right] dz$$

For $\varepsilon > 0$ we will choose a δ such that $z_i, z_j \in \bar{G}_j, \left| f(z) - f(z_j) \right| \leq \varepsilon$ and if $\left| f(z) - f(w) \right| < \varepsilon$ then $|z - w| \leq \delta\sqrt{2}$, since $f \in C(\bar{G})$ we have $z, w \in \bar{G}$. In addition, marking $N = \# \{\text{boundary squares}\}$ and for all G_j we have $L\left(\partial G_j\right) \leq L\left(\partial Q_j\right) + L\left(\partial G \cap Q_j\right)$

$$\left| \int_{\partial G} f dz \right| \leq \sum_{j:Q_j \text{bdry}} \left| \int_{\partial G_j} \left(f(z) - f(z_j) \right) dz \right| < \varepsilon \sum_{j:Q_j \text{bdry}} L(\partial G_j) = \varepsilon \left[4\delta N + L\left(\partial G\right) \right]$$

We will define another number $N_s = \# \left\{ j : Q_j \cap \Gamma_s \right\}$ for all $1 \leq s \leq p$. Clearly, $N \leq \sum_{s=1}^{p} N_s$. We will divide Γ_s into $K = \left[\frac{L(\Gamma_s)}{\delta} \right] + 1$ arcs with equal lengths. Consequently, the length of each arc is $\frac{L(\delta)}{K} < \delta \Rightarrow$ each arc goes through less than 5 squares.

$$N_s \leq 4K \leq d \left[\frac{L(\Gamma_s}{\delta_p} + 1 \right] \Rightarrow$$

$$\begin{aligned} N \leq \sum_{s=1}^{p} N_s \leq 4 \left(\frac{1}{\delta} \sum_{s=1}^{p} L\left(\Gamma_s\right) + p \right) = 4 \left(\frac{L\left(\Gamma\right)}{\delta} + p \right) \\ \leq \varepsilon \left(4 \cdot 4\delta \left(\frac{L\left(\partial G\right)}{\delta} + p \right) + L\left(\partial G\right) \right) \\ = \varepsilon \left(16L\left(\partial G\right) + 16p\delta + L\left(\partial G\right) \right) = \varepsilon \left(17L(\partial G) + 16p \right) \to 0 \end{aligned}$$

□