Complex Function Theory

Mikhail Sodin
Arazim ©

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Today we will perform some integrals.

- Using the known result,

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

We will try to see if the integral is equal over all lines such that $y = const$.Marking the integral $\int_{-\infty}^{\infty} e^{-x^2}$ as $\int_{\mathbb{R}} e^{-x^2}$. Also, we can change the domain of integration to other lines by integrating over $\mathbb{R} + ib$. In these cases we will have, setting $z = x + ib \Rightarrow \dot{z} = 1$.

$$\int_{\mathbb{R}+ib} e^{-z^2} dx = \int_{-\infty}^{\infty} e^{(x+ib)^2} dx = \int_{-\infty}^{\infty} e^{-x^2-2ibx} dx \cdot e^b$$

If we can prove the following equation then we'll be golden:

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot \left(\cos(2bx) + i \cdot \sin(2bx)\right) dx = e^{-b^2} \sqrt{\pi}$$

Since the function is symmetric, the integral of sin is 0. Therefore we can get

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(2bx)\, dx = e^{-b^2} \sqrt{\pi} \Rightarrow \int_{-\infty}^{\infty} e^{-x} \cos(bx)\, dx = e^{--b^2/4\sqrt{\pi}}$$

If we mark a rectangle consisting of the lines $\mathbb{R}$ and $\mathbb{R} + ib$ between the points A and $-A$, with lines from one to the other mark as b and $-b$ respectively. We find that the integral is equal to 0. Proof of correctness:

$$\int_{\gamma_{A,b}} e^{-z^2} dz \xrightarrow{A \to +\infty} 0$$

Of course $z = A + iy, 0 < y < b$.

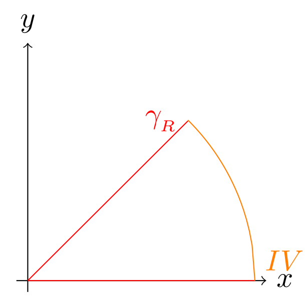
$$\left| \int_{\gamma_{A,b}} e^{-z^2} dz \right| \leq \max_{0 \leq y \leq b} \left| e^{-(A+iy)^2} \right| \cdot b = b \cdot \max_{0 \leq y \leq b} \left[e^{-\Re(A+it)^2} \right] = b \cdot \max_{0 \leq y \leq b} e^{-A^2+y^2} = b \cdot e^{b^2} \cdot e^{-A^2} \xrightarrow{A \to +\infty} 0$$

Where the first inequality comes from $\left| \int_\gamma f dz \right| \leq \max_\gamma |f| \cdot L(\gamma)$

- The second integral we will look at is

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Setting $z = xe^{\pi i/4}$ for $0 \leq x < \infty$ then $\dot{z} = e^{\pi i/4}$ and of course, $z^2 = x^2 e^{\pi i/2} = ix^2$. We will check



$$\int_0^\infty e^{ix^2} e^{\pi i/4} \overset{???}{=} \int_0^\infty e^{-x^2} dx$$

Using the integral

$$\int_0^\infty \sin\left(x^2\right) dx = \int_0^\infty \cos\left(x^2\right) dx = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Then we have

$$\int_0^\infty e^{-ix^2} dx = \int_0^\infty \cos\left(x^2\right) - i \sin\left(x^2\right) dx = e^{\pi i/4} \frac{\sqrt{\pi}}{2}$$

Setting γ_R as the arc from 0 to $\pi/4$ with a radius of $R >> 1$ (eighth of a circle) and Γ_R as the closed arc connecting to the origin (looks like a slice of pizza), then

$$\int_{\Gamma_R} e^{-z^2} dz = 0 \qquad \int_{\gamma_R} e^{-z^2} dz \xrightarrow[R\to\infty]{???} 0$$

In γ_R we have $z = Re^{i\theta}$, $0 \leq \theta \leq \frac{\pi}{4}$, $\dot{z} = iRe^{i\theta}$ Then:

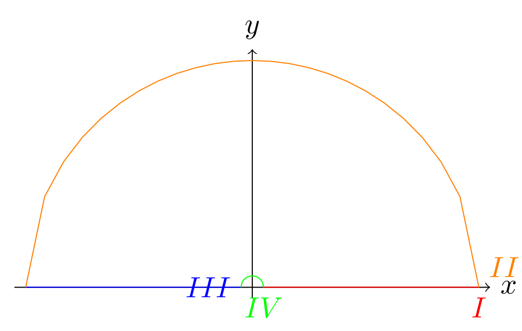
$$\begin{aligned}
\left| \int_0^{\pi/4} e^{-R^2} e^{2i\theta} iRe^{i\theta} d\theta \right| &\leq R \cdot \int_0^{\pi/4} e^{-R^2 \cos(2\theta)} d\theta = \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-R^2 \cos\theta} d\theta = \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-R^2 \sin\theta} d\theta \\
&\leq \frac{R}{2} \frac{0}{\pi/2} e^{-R^2 \cdot \frac{2\theta}{\pi}} d\theta = \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi} R^2 \theta} d\theta \\
&< \frac{R}{2} \int_0^\infty e^{-\frac{2}{\pi} R^2 \theta} d\theta = \frac{R}{2} \cdot \frac{\pi}{2R^2} = \frac{\pi}{4R} \xrightarrow{R\to\infty}
\end{aligned}$$

- The third integral is

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} = \frac{\pi}{2}$$

Setting $0 < \varepsilon < R < \infty$ and $\varepsilon \downarrow 0$, $R \uparrow \infty$ then

$$\frac{e^{iz}}{z} = \frac{1}{z} \left(1 + \sum_{n \geq 1} \frac{i^n z^n}{n!} \right) \overset{\star}{=} \frac{1}{z} + \sum_{n \geq 0} \frac{i^{n+1}}{(n+1)!} z^n$$



We will define the following arcs:

I is from ε to R on the real line. II is the half circle with a radius of R and a positive imaginary part.
III is from $-R$ to ε on the real line.
IV is a half circle with a radius of ε.

$$0 = \int_{\Gamma_{\varepsilon,R}} \frac{dz}{z} \overset{\star}{=} \int_{\Gamma_{\varepsilon,R}} \frac{e^{iz}}{z} dz \overset{\star}{=} \int_1^z dz = \cdots = \int_{\text{I+III}} + \underbrace{\int_{II}}_{\to 0} + \underbrace{\int_{\text{iIII}}}_{-\pi i}$$

$$\int_{\mathrm{I+III}} = \int_{\varepsilon}^{R} \frac{e^{ix}}{x} dx - \int_{\varepsilon}^{R} \frac{e^{ix}}{x} = 2i \cdot \int_{\varepsilon}^{R} \frac{\sin x}{x} dx$$

$$\int_{IV} \frac{e^{iz}}{z} = \int_{IV} \frac{dz}{z} + \underbrace{\int_{IV} \sum (\dots)}_{|\gamma_{IV} = \pi\varepsilon|} \cong -\int_{0}^{\pi} \frac{\varepsilon i e^{i\theta}}{\varepsilon e^{i\theta}} d\theta = -\pi i$$

It is sufficient to prove that $\int_{II} (\dots) \to 0$ and then when $R \to \infty, \varepsilon \to 0$:

$$2i \int_{0}^{\infty} \frac{\sin x}{x} dx - \pi i = 0 \Rightarrow \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

And in a similar fashion if we integrate

$$\int_{\Gamma_{\varepsilon,R}} \frac{1 - e^{iz}}{z^2} \Rightarrow \int_{0}^{\infty} \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}$$