Complex Function Theory

Mikhail Sodin
Arazim ©

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1 Complex numbers

Numbers that can be written as $z = x + yi, x, y \in \mathbb{R}, i^2 = -1$

1. Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
2. Multiplication: $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ $z \neq 0(= 0 + 0i), z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

Field: $\mathbb{R} \hookrightarrow \mathbb{C}$ $x \mapsto x + i0$
Conjugate:

$$z = x + yi,\ \bar{z} := x - yi \qquad x = \frac{z + \bar{z}}{2}, y = \frac{z + \bar{z}}{2i} \qquad z \cdot \bar{z} = x^2 + y^2 = |z|^2$$

,

$$|z| := \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \geq 0 \qquad \overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}, \overline{z_1 \cdot z_2} = \bar{z_1} \cdot \bar{z_2}$$

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = z_1 \cdot \bar{z_1} + z_1 \cdot \bar{z_2} + z_2 \cdot \bar{z_1} + z_2 \cdot \bar{z_2} = |z_1|^2 + 2\Re(z_1 \bar{z_2}) + |z_2|^2$$

1.1 Inequalities

1. Triangle: $|z_1 + z_2| \leq |z_1| + |z_2|$

   Proof. $|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1 \bar{z_2}| = |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z_2}| = \left(|z_1| + |z_2|\right)^2$ □

2. Cauchy-Schwartz:

$$\left|\sum_j z_j w_j\right|^2 \leq \sum_j |z_j|^2 \cdot \sum_k |w_k|^2$$

   When is there an equality in (1) and (2).

1.2 Gaussian Plane

This plane consists of two axis, the real numbers and the complex. A number $z \in \mathbb{C}$ is written as $z = r\cos\phi + ir\sin\phi = r(\cos\varphi + i\sin\varphi)$ φ is sometimes referred to as the **argument** of z. It is easy to see that multiplication and addition work as before. As a result, $z^n = r^n(\cos\varphi + i\sin\varphi)$. Possible solutions to the equation $z^n = a, a \in \mathbb{C}\setminus\{0\}, n \in \mathbb{N}\setminus\{1\}$

$$\text{Moivre} \Rightarrow \begin{cases} |z|^n = |a| & \rightarrow r = |z| = |a|^{1/n} \\ r \cdot \arg(z) & = \arg(a) \end{cases}$$

$$\varphi_k = \frac{\theta}{-} + \frac{2\pi k}{n}, k = 0, 1, \ldots, n-1 \qquad r = |a|^{1/n}$$

2 Topology in $\mathbb{C} = \mathbb{R}^2$

$\rho(z, w) := |z - w|$ (Euclidian distance).

- Convergence: $z_n \to z$ if $|z_n - z| \to 0$ ($\Leftrightarrow \Re(z_n) \to \Re(z), \Im(y_n) \to \Im(y)$).
- Open Circle: $\mathcal{D}(a, r) = \{z_1 : |z - a| < r\}$
  Closed: $\bar{\mathcal{D}}(a, r) = \{z_1 : |z - a| \leq r\}$
- $A \subset \mathbb{C}$ z is an interior point if there exists a ball $\mathcal{D} \subset A$ $int(A) = interior =$ all of the interior points of A.
- $A \subset \mathbb{C}$ z is an exterior point if there exists a ball $\mathcal{D}\left(z, r)\right) \cap A = \emptyset$ $ext(A) = exterior =$ all of the exterior points of A.
- $\mathbb{C} = int(A) \uplus ext(A) \uplus \partial A$ (The intersection is empty).
- A is open if $A = int(A)$.
- A is closed if $\mathbb{C} \backslash A$ is open.

2.1 Compact sets

Lemma/Definition: $K \subset \mathbb{C}$ then the following are equivalent:

1. Heine-Borel: For every covering of K by open sets there exists a finite subcovering.
2. For every series $\{z_n\} \subset K$ there exists a subsequence $\left\{z_{n_j}\right\}$ which converges and $\lim_{n_j} z_{n_j} \in K$
3. K is closed and bounded.

$$\rho(A, B) := \inf_{a \in A, b \in B} \rho(a, b)$$

2.2 Connectivity

$G \subset \mathbb{C}$ is an open set.

Definition 1. G is polygonally connected if for all $z, w \in G$ there exists a polygonal line $L \subset G$ with a staring point z and a finish line w.

Definition 2. Let $f : G \to \mathbb{C}$. We will say that f is locally constant if for every point $z \in G$ there exists an open set $z \in O \subset G$ such that $f|_O = const$.

Lemma 1. *$G \subset \mathbb{C}$ is an open set. The following are equivalent:*

1. *G is polygonally connected.*
2. *If $X \subset G$ is an open set and $G \backslash X$ is open then $X = G$ or $X = \emptyset$.*
3. *If $f : G \to \mathbb{C}$ is locally constant then $f \equiv const$.*

(2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2). (2) $\Rightarrow$ (1) $z \in G$

$$X := \{w \in G : \exists \text{polygonal line in } G \text{ from } z \text{ to } w\}$$

Using the fact that both X and $G \backslash X$ are open, and since $X \neq \emptyset$(since $z \in G$) we can arrive at the fact that $X = G$.
(1) $\Rightarrow$ (3)

Lemma 2. *$I \subset \mathbb{R}$ is a closed set (and finite), $f : I \to \mathbb{R}$ is locally constant then $f \equiv const$*

(3) $\Rightarrow$ (2) We will define

$$f(z) = \chi_X(z) = \begin{cases} 1 & z \in X \\ 0 & z \in G \backslash X \end{cases}$$

f is locally constant then by (3) we have that either $f \equiv 1, X = G$ or $\equiv 0, X = \emptyset$ $\square$

Definition 3. $G \subset \mathbb{C}$ is open, G is a domain (region) if the qualities of the lemma apply.