Complex function theory - recitation

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1 Residues

If f is analytic in $0 < |z - z_0| < R$ and in that domain,

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

For

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$(0 < \rho < R.)$ Then $res(f, z_0) = res_{z_0} f = a_{-1}$

Theorem 1 (Cauchy)**.** *Let G be a good domain and $f \in Hol\left(G \setminus \{a_1, a_2, \ldots, a_n\}\right)$ then*

$$\frac{1}{2\pi i} \int_{\partial G} f = \sum_{a_j \in G} res_{a_j} f$$

1.1 Finding the residue

1. If f has a simple pole at z_0 then

$$res_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z)$$

2. If f has a pole with an order of m at z_0 then there exists a ψ which is analytic and different from z_0 such that $f(z) = \frac{\psi(z)}{(z-z_0)^m}$ and the following holds:

$$res_{z_0} f = \frac{\psi^{(n-1)(z_0)}}{(n-1)!}$$

Problem 1. Find the Laurent expansion for

$$f(z) = \frac{z - 7}{z^2 - z - 2}$$

around the following rings:

1. $|z| < 1$.
2. $1 < |z| < 2$.
3. $|z| > 2$.

Solution 1. We will define

$$f(z) = \frac{3}{z+2} - \frac{2}{z-1} = f_1 + f_2$$

$$f_1(z) = \begin{cases} \frac{3}{2}\sum_{n\geq 0}\left(\frac{-z}{2}\right)^n & |z|<2 \\ \frac{3}{2}\sum_{n\geq 0}\left(\frac{2}{z}\right)^n & |z|>2 \end{cases} \qquad f_2 = \begin{cases} e\sum_{n\geq 0} z^n & |z|<1 \\ -\frac{2}{z}\sum_{n\geq 0}\left(\frac{1}{z}\right)^n & |z|>1 \end{cases}$$

Problem 2. Categorize the points of singularity of the following points and calculate the residues.

1. $$f(z) = z^2 \sin\frac{1}{z}$$

2. $$g(z) = \frac{\sin z}{z(z-1)^2}$$

Solution 2.

1. $z=0$ is the lone point of singularity. Using the fact that

$$\sin w = \sum_{n\geq 0} \frac{(-1)^n w^{2n+1}}{(2n+1)!}$$

Thus,

$$z^2 \sin\frac{1}{z} = z^2 \sum \frac{(-1)^n z^{-(2n+1)}}{(2n+1)!}$$

This is an essential singularity (infinite number of negative powers) and $res_0 f = \frac{-1}{6}$.

2. There are points of singularity at $z=1, z=0$, $\lim_{z\to 0} g(z) = 1$, thus 0 is a removable singularity (residue zero). We will define $\psi(z) = \frac{\sin z}{z}$, then $g(z) = \frac{\psi(z)}{(z-1)^2}$ and ψ is analytic at 1. $\psi(1) \neq 0$ thus 1 is a pole with an order of 2 and

$$res_1 g = \frac{\psi'(1)}{1!} = \cos 1 - \sin 1$$

Problem 3. Find

$$\int_{|z|=1} \sin\left(\frac{1}{z}\right) \cdot \sin(z)\, dz$$

Solution 3. We will define $f(z) = \sin\left(\frac{1}{z}\right) \cdot \sin(z)$ and we will want to find $res_0 f$. f is even, and as a result the Laurent expansion has only even powers and the residue is 0.

Problem 4. Let p be a polynomial with simple zeroes only ($a_1, a_2, \ldots, a_n$ are the zeroes), show that

$$\sum_{a_i} \frac{1}{p'(a)} = 0$$

Solution 4. We will define $g(z) = \frac{1}{p}$, g is analytic at every point other that $a_1, a_2, \ldots, a_n$ (removable singularity at ∞) and at $a_1, a_2, \ldots, a_n$ we have poles.

$$res_{a_j} g = \lim_{z\to a_j} \frac{(z-a_j)}{p(z)} \overset{\text{L'hospital}}{=} \frac{1}{p'(a_j)}$$

Thus by the Cauchy theorem we arrive at

$$\sum_{1\leq j\leq n} \frac{1}{p(a_j)} = \frac{1}{2\pi i}\int_{|z|>\max a_j} g(z)dz = 0$$

Since ∞ is a removable singularity.