Complex function theory - recitation

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1 Uniqueness thoerems

Theorem 1. *Let $f \in Hol(G)$ and $a \in G$, then:*

1. *If $f^{(n)}(a) = 0$ for all $n \geq 0$ then $f \equiv 0$.*
2. *If a sequence $a \neq z_n \to a$ in G, $f(z_n) = 0$ then $f \equiv 0$.*

Definition 1 (Order of 0)**.** f is a 0 of order m at a if $m := \min\left\{n \in \mathbb{Z}_+ : f^{(n)}(a) = 0\right\}$
For an analytic function at a there is a 0 of order m iff $f(z) = (z-a)^m \cdot g(z)$ where g is analytic at a and $g(a) \neq 0$.

Problem 1. Let $f(z) = \sum_{n \geq 0} a_n z^n$ which is analytic in $|z| < 1$. Show that $\left|f(\frac{1}{n})\right| \leq 2^{-n}$ for all n $\Rightarrow f \equiv 0$.

Solution 1. f is continuous therefore $f(0) = 0$ ($a_0 = 0$). Assume that we have shown that $a_{k-1}, a_{k-2}, \ldots, a_0 = 0$, we will show that $a_k = 0$.

$$f(z) = \sum_{n=k}^{\infty} a_n z^n = z^k \overbrace{\sum_{n=0}^{\infty} a_{n+k} z^n}^{g(z)}$$

$$g(z) = z^{-k} \cdot f(z) \qquad \left|g(\frac{1}{n}\right| = \left|n^k f\left(\frac{1}{n}\right)\right| \leq n^k 2^{-n} \xrightarrow[n\to\infty]{} 0 \Rightarrow g(0) = 0 \Rightarrow a_k = 0$$

2 Laurent series

f is analytic in $t <|z-a| < R$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n = \overbrace{\sum_{n=-\infty}^{-1} a_n (z-a)^n}^{\text{Singular}} + \overbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}^{\text{Analytic}}$$

Where every a_n is defined for all $n \in \mathbb{Z}$ as

$$a_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz$$

For all $r < \rho < R$. When $r = 0$ we will say that a is an isolated singularity.

2.1 Categorizing of singular points

Let f be analytic in $0|z-a|<R$, a is a singular point.

1. If the limit $\lim_{z\to a} f$ exists and is finite then a is a removable singularity.
2. If $\lim_{z\to a} f = \infty$ then f is a pole and the order of the pole is m then $\sum_{n=-m}^{-1} a_n (z-a)^n$ is the singular part of the series.
3. If the limit $\lim_{z\to a} f(z)$ doesn't exist then a is an essential singularity.

Problem 2. Let f be entire. Assume that $\lim_{z\to\infty} f(z) = \infty$ show that f is a polynomial.

Solution 2. Proved in lecture 15.
ANother solution for this is to say that there exists an M such that for all $|z|>M$, $\left|f(z)\right| \geq 1$. In the set $|z| \leq M$ there is a finite number of zeroes (else according to the uniqueness theorem $f \equiv 0$). We will mark the zeroes of f as $a_1, a_2, \dots, a_n$ including the multiplicity.

$$g(z) = \frac{f(z)}{(z-a_1)(z-a_2)\cdots(z-a_n)}$$

g has removable singularity at $a_1, a_2, \dots, a_n$ meaning that g is entire (we can expand g to be entire) and $g(z) \neq 0$ for all z. In addition $h(z) = \frac{1}{g(z)}$ is whole and $h \neq 0$. For $|z| > M$ we have $\left|\frac{1}{f}\right| \leq 1$. By the triangle inequality $|z| \geq M$, $\left(|z|+M\right)^k \geq \left|h(z)\right|$.
The set $\left\{|z| \leq M\right\}$ is compact and therefore there exists a $C>0$ such that for all $|z| \leq M$, $\left|h(z)\right| \leq C$. In total we have $\left|h(z)\right| \leq \left(|z|+M\right)^k + C$. h is entire and therefore from last week is a polynomial of degree at most k. Since h has no zeroes, according to the fundamental theorem of algebra $h \equiv const$, thus f is a polynomial that vanishes at $a_1, a_2, \dots, a_n$.

Problem 3. Let $f \in Hol(\mathbb{C} \setminus \{0\})$ and assume that there exists an $a \in \mathbb{R} \setminus \mathbb{Z}$. We will assume that for all r,

$$\int_0^{2\pi} \left| f\left(er^{i\theta}\right)\right| d\theta \leq r^a$$

Show that $f \equiv 0$

Solution 3. $f(z) = \sum_{n=\infty}^{\infty} a_n z^n$ and show that $a_n = 0$ for all $n \in \mathbb{Z}$

$$|a_n| = \left|\frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}} dz\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\left|f(re^{i\theta})\right|}{\left|r^{n+1}e^{i(n+1)\theta}\right|}\left|ire^{i\theta}\right| d\theta \leq \frac{1}{2\pi r^n}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right| d\theta \leq \cdots \leq \frac{r^{\alpha-n}}{2\pi}$$

If $\alpha > n$ then as $r \to 0$, $a_n \to 0$ and if $\alpha < n$ then as $r \to \infty$, $a_n \to 0$.