Complex function theory - recitation

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1 Cauchy theorem/equation and its uses

Theorem 1 (Cauchy theorem). *Let G be a good domain and $f \in Hol\left(G\right) \cap C\left(\bar{G}\right)$. Then*

$$\int_{\partial G} f(z)dz = 0$$

Theorem 2 (Cauchy equation).

$$\frac{1}{2\pi i}\int_{\partial G}\frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0) & z_0 \in G \\ 0 & z_0 \in \mathbb{C}\backslash G \end{cases}$$

One can derivate under the integral sign ∞ times.

$$\frac{n!}{2\pi i}\int_{\partial G}\frac{f(z)}{(z-z_0)^{n+1}} = \begin{cases} f^{(n)}(z_0) & z_0 \in G \\ 0 & z_0 \in \mathbb{C}\backslash G \end{cases}$$

Corollary 1.
1. *The maximum principle: If $f \in Hol(G)$ and $|f|$ has a local maximum in G, then f is constant.*
2. *Liouville theorem: If f is an entire function ($Hol(\mathbb{C})$) and bounded then f is constant.*

Problem 1. Show that if f, g are to golomorphic functions in a bounded domain G and continuous up to the boundary, then the maximum of $|f| + |g|$ is at the boundary.

Solution 1. We will mark with $z_0 \in \bar{G}$ the point at which the maximum of $|f| + |g|$ is arrived at(from weierstrauss). There exist $|C_1| = |C_2| = 1$ such that $c_1 f(z_0) = \left|f(z_0)\right|$ and $C_2 g(z_0) = \left|g(z_0)\right|$. We will define $h = C_1 f + C_2 g$. By the triangle inequality, $|h| \leq |g| + |f|$ and on the other hand,

$$\left|h(z_0)\right| = \left|f(z_0)\right| + \left|g(z_0)\right| \geq \left|f(z)\right| + \left|g(z)\right|$$

Since h is analytic in G, we have two options,

1. $z_0 \in G$ and h is constant and we are done.
2. !!!!

Problem 2. $D = \left\{z : |z| < 1\right\}$. Given $f \in Hol(D) \cap C(\bar{D})$. Assume that

$$\left|f(z)\right| \leq \begin{cases} 3 & \Re z > 0 \\ 2 & \Re z < 0 \end{cases}$$

Show that

1. $\left|f(0)\right| \leq \frac{5}{2}$.

2. $\left|f(0)\right| \leq \sqrt{6}$.

Solution 2. 1.

$$f(0) = \frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)}{z}dz$$

$$\left|f(0)\right| \leq \frac{1}{2\pi}\int_0^{2\pi}\left|f\left(e^{it}\right)\right|dt \leq \frac{3+2}{2} = \frac{5}{2}$$

2. We will define $g(z) = f(z)\cdot f(-z)$ whcih is analytic in D and continuous up to the boundary. $|g| \leq 6$ for $z \in \bar{D}\backslash i\mathbb{R}$

$$\left|f(0)\right|^2 = \left|g(0)\right| = \left|\frac{1}{2\pi i}\int_{|z|=1}\frac{g(z)}{z}dz\right| \leq 6$$

Problem 3. Let f be an entire function, assume that $\Re f$ is bounded from above. Show that f is constant.

Solution 3. We will define $g = e^f$. $|g| - e^{\Re f} < \infty$ and then g is constant, $0 \equiv g' \bigcap f' \cdot e^f$ and since $e^f \neq 0$ we have $f' \equiv 0$.