Complex function theory - recitation

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1 Power series and integrals

Problem 1. For the following power series find a_n and calculate the convergence radius:

$$\log^2(1-z) = \sum a_n z^n$$

Solution 1. The taylor series of $\log(1-z) = \sum \frac{z^n}{n}$, thus our series is this squared. Using the formula for multiplication of two series:

$$\left(\sum_{n\geq 0} b_n z^n\right)\left(\sum_{n\geq 0} c_n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_k c_{n-k}\right) z^n$$

In our case we have

$$\log^2(1-z) = \left(\sum_{n\geq 1}\frac{z^n}{n}\right)\left(\sum_{n\geq 1}\frac{z^n}{n}\right) = \sum_{n\geq 2}\left(\sum_{k=1}^{n-1}\frac{1}{n-k}\frac{1}{k}\right) z^n$$

If we develop $\frac{1}{n-k}\frac{1}{k} = \frac{1}{n}\left(\frac{1}{n-k}+\frac{1}{k}\right)$. By symmetry this is equal to $\frac{1}{n}\cdot 2\cdot\frac{1}{k}$ Giving us:

$$\log^2(1-z) = \sum_{n=2}^{\infty}\frac{2}{n}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right) z^n = 2\sum_{n=2}^{\infty}\frac{H_{n-1}}{n} z^n$$

Where H_{n-1} is the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}$. We will now find the convergence radius using the Cauchy-Hadamard equation, $\frac{1}{R} = \overline{\lim}|a_n|^{1/n}$:

$$\frac{1}{n} \leq \frac{H_{n-1}}{n} \leq 1$$

$$\left(\frac{1}{n}\right)^{1/n} \leq \left(\frac{H_{n-1}}{n}\right)^{1/n} \leq 1$$

And this gives us $\left(\frac{H_{n-1}}{n}\right)^{1/n} \to 1$ meaning that the convergence radius is 1. □

Integrals

Before we begin the next question, some defintions:
For a real ($\in \mathbb{R}$) segment $[a, b]$ and a function $f : [a, b] \to \mathbb{C}$ we have

$$\int_a^b f(z)dz = \int_a^b \Re\big(f(x)\big)\,dx + i\int_a^b \Im\big(f(x)\big)\,dx$$

For an arc $\gamma : [a, b] \to \mathbb{C}$ we have

$$\int_\gamma f(z)dz = \int_a^b f\big(\gamma(t)\big)\,\gamma'(t)\,dt$$

If f is analytic on $\gamma : [a, b] \to \mathbb{C}$ then

$$\int_\gamma f'(z)dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big)$$

$$\int_{|z|=R} z^n dz = \begin{cases} 1 & n \neq -1 \\ n = 2\pi i & \end{cases}$$

Problem 2. Show that

$$\int_0^\infty \frac{1-\cos x}{x^2} = \frac{\pi}{2}$$

Solution 2. We will define $f(z) = \frac{1-e^{iz}}{z^2}$. For $R > \varepsilon > 0$ we will define:

- $\Gamma_1 = [\varepsilon, R]$
- $\Gamma_R = \{z||z| = R \wedge \Im z > 0\}$ (a half circle).
- $\Gamma_2 = [-R, \varepsilon]$
- $\Gamma_\varepsilon = \{z||z| = \varepsilon \wedge \Im z > 0\}$ (a half circle).
- $\Gamma_{\varepsilon,R}$ the whole thing.

Using the theorem proved in class that

$$\left|\int_\gamma f\right| \leq L(\gamma)\max_\gamma|f|$$

We will show that

$$\int_{\Gamma_{\varepsilon,R}} f(z) = 0$$

Since f has an antiderivative,

$$\frac{1}{z^2}\left(1 - \sum_{n\geq 0}\frac{(in)^n}{n!}\right) = \frac{1}{z} + \sum_{n=2}^{\infty}\sum_{z^{n-2}}^{n!}$$

$$\int_{\Gamma_1} fdz + \int_{\Gamma_2} fdz = \int_\varepsilon^R \frac{1-e^{ix}}{x^2} + \frac{1-e^{ix}}{x^2}dx = 2\int_\varepsilon^R \frac{1-\cos x}{x^2}$$