Complex function theory - recitation

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Definition 1. A **Conformal map** f is $f : \mathbb{C} \to \mathbb{C}$ one to one and holomorphic (in particular $f' \neq 0$)

A common marking is $\bar{\mathbb{C}} = \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$

Definition 2. The transformation $f(z) = \frac{az+b}{cz+d}$ is called a möbius transformation and has the follwing properties:

1. f is möbius $\Rightarrow$ f is bi-holomorphic (holomorphic, invertible and f^{-1} is holomorphic). $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (and every such tranformation is möbius).
2. Let $\{z_1, z_2, z_3\} \subset \hat{\mathbb{C}}$ and $\{z_1, z_2, z_3\} \subset \hat{\mathbb{C}}$ then there xists a single mobius tranformation such that $f(z_i)\, w_i$ given by $w = f(z)$, $(w_1, w_2, w_3, w) = (z_1, z_2, z_3, z_4)$ where

$$(a, b, c, d) := \left(\frac{a-c}{a-d}\right) / \left(\frac{b-c}{b-d}\right)$$

Definition 3 (Generalized circle)**.** A generalized circle is acircle or a line. 3 points define a single generalized circle.

Claim 1. z, z^* are symmetric in relation to a generalized circle C iff there exist $z_1, z_2, z_3 \in C$ such that $(z_1, z_2, z_3, z^*) = \overline{(z_1, z_2, z_3, z)} \Leftrightarrow$ for any 3 points in C.

Claim 2. Let $f \in M\ddot{o}b\left(\hat{\mathbb{C}}\right)$, z, z^* are symmetric with relation to C, then $f(z), f(z^*)$ are symmetric with relation to $f(C)$.

Problem 1. Let $R > 0$. Find a $f \in M\ddot{o}b\,(\mathbb{C}_+)\,, f : \mathbb{C}_+ \to \big\{w \big| |w| < R\big\}$ where $\mathbb{C}_+$ is defined as $\big\{z \in \mathbb{C} \big| \Im(z) > 0\big\}$ such that $f(\alpha) = 0$.

Solution 1. $f(z) = \frac{az+b}{cz+d}$ and by symmetry, $f(\alpha) = 0$, $f(\bar{\alpha}) = \infty$ since the symmetric point of 0 according to $B(0, R)$ is ∞. We have $a\alpha + b = 0$ and $c\bar{\alpha} + d = 0$ then we can write $f(z) = \lambda \frac{z-\alpha}{z-\bar{\alpha}}$.
$f(o) = \lambda \cdot \frac{-\alpha}{-\bar{\alpha}} = \lambda \cdot \frac{\alpha^2}{|\alpha|^2}\, \big|f(0)\big| = R, \big|f(0)\big| = |\lambda|$ then $|\lambda| = R$ and we have found

$$f(z) = Re^{i\theta} \frac{z+\alpha}{z-\alpha}$$

Problem 2. Find $f(T)$ given $f(z) = \frac{1}{z}$ for the following T.

1. $T = \big\{z \in \mathbb{C} \big| |z - i| = 1\big\}$
2. $T = \left\{z \in \mathbb{C} \big| |z - i| = \sqrt{2}\right\}$

Solution 2. We will look at three points in each circle and see where they point to, since three points define a circle.

1. $f(0)=\infty$ $f(2)=\frac{1}{2}$ and $f(1+i)=\frac{1}{1+i}=\frac{i-i}{2}$ the only line which goes through all three of these points is $f(T)=\left\{w|\Re\left(w\right)=\frac{1}{2}\right\}$.

2. $f(1+\sqrt{2})=\frac{1}{1+\sqrt{2}}$, $f(-i)=i$ and $f(i)=-i$ meaning that the tranformation inverts the circle around the real axis.