Complex function theory - recitation

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1 Cauchy-Riemman equations

$f = u + iv$ where $u, v : G \to \mathbb{R}$ and $G \subseteq \mathbb{R}^2 = \mathbb{C}$ then

$$u_x = v_y$$
$$u_y = -v_y$$

Theorem 1. *If G is a domain then*

1. *u, v holf the Cauchy Riemman equation $\Rightarrow f \in H(G)$.*
2. *$f \in H(G) \Rightarrow u, v$ hold the C-R equations.*

Problem 1. D is a domain, $f, g \in H(D)$ and $\Re f = \Im g$ in D. Show that there exists a $c \in \mathbb{C}$ such that $g = if + c$.

Solution 1. We will define a $h = g - if, h \in H(d), \Im h \equiv 0$. Thus, according to a theorem that we have learned, $h \equiv c$.

Problem 2. Let $f = u + iv, f \in H(\mathbb{C})$ given that $v = g(u)$ where $g : \mathbb{R} =\!\!\rightarrow \mathbb{R}$ is differentiable. Show that f is constant.

Solution 2. From the chain rule and the C-R equations, we have

$$\begin{matrix} -u_y = v_x = g'(u) \cdot u_x \\ u_x = v_y = g'(u) \cdot u_y \end{matrix} \Rightarrow \begin{matrix} g'(u)u_x + u_y = 0 \\ -u_x + g'(u)u_y = 0 \end{matrix} \Rightarrow u_x \equiv u_y \equiv 0$$

Thus u is constant and as a result, f is too.

2 Harmonic functions

Definition 1. $U : G \to \mathbb{R}, u \in C^2(G)$ we will sat that $u \in Harm(G)$ iff $\triangle u \equiv 0$ in G $(\triangle u = u_{xx} u_{yy})$

Theorem 2. *If $f = u + iv$ is analytic in G, $u, v \in C^2(G)$ then u, v is harmonic.*

Definition 2. If $F : G \to \mathbb{C}$ is anayltic and $f = u + iv$ where u, v are harmonic then u, v are called **conjugate harmonic**.[1]

Theorem 3. *If G is $\mathbb{C}$ or a disk or a rectangle that is parallel to the axes then for all harmonic function u there is a conjugate in G.*

Problem 3. When is the polynomial $p(x, y) = ax^2 + bxy + cy^2$ a harmonic function? $(x, y, a, b, c \in \mathbb{R})$

[1] Given the aforementioned u, its harmonic conjugate v is unique up to a constant (given that it exists).

Solution 3.

$$o \stackrel{?}{=} p_{xx} + p_{yy} = 2a + 2c \Rightarrow a = -c$$

$$p(x,y) = a(x^2 - y^2) + b \cdot xy \stackrel{x=x+iy}{=} a \cdot \Re(z^2) + \frac{b}{2}\Im\left(z^2\right)$$

Problem 4. Show that the function $u(x,y) = u(z) = \log|z|$

1. Is harmonic $\mathbb{C}\setminus\{0\}$
2. u doesn't have a conjugate in this domain.

Solution 4. 1. $u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}(1) + 0 = 0$ and consequently, u is harmonic in $\mathbb{C}\setminus\{0\}$.

2. We will assume by contradiction that u has a conjugate v in the domain. Using the polar coordinate form of the C-R equations

$$\begin{array}{ll} (1) & u_r = \frac{1}{r}v_\theta \\ (2) & v_r = -\frac{1}{r}u_r \end{array}$$

Then $v_\theta = 1$ and $v_r = 0$, thus $v(r,\theta) = h\theta$, $h'(\theta) = 1$ meaning that

$$v(r,\theta+2\pi) - v(r,\theta) = \int_{\theta+2\pi}^{\theta} 1dr = 2\pi$$

And as a result, v is not continuous.

Reminder 1.

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial z} - i\frac{\partial}{\partial y}\right)$$

$$\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial z} + i\frac{\partial}{\partial y}\right)$$

Problem 5. Let $f : G \to \mathbb{C}$ be a holomorphic function and $f \neq 0$ in G. Show that $u = \log|f|$ is a harmonic function.

Solution 5. It is sufficient to show that $2u = \log|f|^2$ is harmonic.

$$\triangle(2u) = 4\partial_z\partial_{\bar{z}}\log|f|^2 = 4\partial_z \cdot \partial_{\bar{z}}\log f \cdot \bar{f} = 4\partial_z\left(\frac{\partial_{\bar{z}}(f\bar{f})}{f \cdot \bar{f}}\right) = 4\partial_z\left(\frac{\partial_{\bar{z}}f \cdot \bar{f} + f\partial_{\bar{z}}\bar{f}}{f\bar{f}}\right) \stackrel{\partial_{\bar{z}}=\overline{\partial_z \bar{f}}}{=} \cdots = 0$$