Complex function theory - recitation

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1 A little topology

Definition 1. A set $U \subseteq \mathbb{C}$ is called an open set if for all $z \in U$ there exists a disk D such that $z \in D \subseteq U$

Definition 2. $X \subseteq \mathbb{C}$ is connected iff there fo not exists open and disjoint sets $A, B \subseteq U$ such that $X = A \cup B$.

Definition 3 (Polygonal/Arc connectivity)**.** $X \subseteq \mathbb{C}$ is called polygonally/arc connected if for all $z_1, z_2 \in X$ there exists a polygonal line/ arc which is a subset of X and connects z_1 and z_2.

Definition 4 (Domain)**.** An open and connected set is called a domain.

Problem 1. Show that a domain is polygonally connected.

Solution 1. Let X be a domain and $z_0 \in X$ we will define

$$A = \{z \in X : \text{There exists a line between } z_0 \text{ and } z\}$$

$$B = \{z \in X : \text{No line exists between } z_0 \text{ and } z\}$$

$X = A \cup B$ we will assume that X is not polygonally connected. Thus $B \neq \emptyset$ and $z_0 \in A \neq \emptyset$. Consequently, a;; that is left to show is that A, B are open and we will arrive at a contradiction to the fact that X is conencted.
We will show that X is open: Let $z' \in A$ Since X is open there exists a disk $z' \in D \subseteq X$ and by definition of A there isa polygonal line Γ which connects z_0 with z' for $z \in D$ the polygonal line $\Gamma \cup [z', z]$ connects z_0 with z. Thus $D \subseteq A$ and A is open. In a similar fashion, B is open.

2 Limits and continuity

Definition 5. A sequence $z_n \to z_0$ iff $|z_n \to z_0| \to 0$.

Problem 2. Check the convergence of the following sequences:

1. $a_n = n \cdot z^n \qquad z \in \mathbb{C}$
2. $\arg\left(\frac{i-n}{2n-i}\right) \qquad \arg \in [-\pi, \pi)$

Solution 2. 1.

$$|a_n| = n \cdot |z|^n \to \begin{cases} \infty & |z| \geq 1 \\ 0 & |z| < 1 \end{cases}$$

2.

$$\frac{i-n}{2n-i} = \frac{(i-n)(2n+i)}{4n^2+1} = \overbrace{\frac{-2n^2-1}{4n^2+1}}^{\to -1/2} + i\overbrace{\frac{n}{4n^2+1}}^{\to 0^+}$$

and thus,

$$b_n = \arctan\left(\frac{\frac{n}{4n^2+1}}{\frac{-2n^2-1}{4n^2+1}}\right) + \pi = \pi$$

Definition 6. $f : \mathbb{C} \to \mathbb{C}$ is differentiable at the point z if the following complex limit exists:

$$f'(z) = \lim_{\mathbb{C}\ni h\to 0} \frac{f(z-h)-f(z)}{h}$$

Definition 7. A function f which is differentiable in a domain Ω is called analytic or holomorphic in Ω.

Theorem 1. *If $f' \equiv 0$ in Ω then f is constant in Ω.*

Problem 3. Let $f = u_i v$ be an analytic gunction in the somain Ω show that if $u \equiv c$ for some constant c in Ω then f is constant.

Solution 3. See lecture 3.

Corollary 1. *$f(z) = |z|$ is not analytic in any domain on the complex plane.*

Problem 4. Let $g : [0,1] \to \mathbb{C}$ be a continuous function in $[0,1]$. We will define

$$F(Z) = \int_1^0 \frac{g(t)}{z-t} dt$$

Show that F is analytic in the domain $\mathbb{C}\backslash[0,1]$.

Solution 4. For $z, z_0 \in \mathbb{C}\backslash\, [0,1]$

$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\left[\int_0^1 \frac{g(t)}{z-t}dt - \int_0^1 \frac{g(t)}{z_0-t}dt\right] = \int_0^1 \frac{(z_0-z)\,g(t)}{(z-t)(z_0-t)}dt$$