Complex function theory - recitation

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1 Schwarz's reflection principle

Version a Let $v \; in Harm(\Omega_+)$ and continuous up to the boundary, $v|_I = 0$ then v has a harmonic continuation to Ω_1 where $v(z) = -v(\bar{z})$.

Version b Let $f \in Hol(\Omega)$ and $\Im f \xrightarrow[z\in\Omega_+]{z\to x\in I} 0$ then f has a analytic continuation in Ω where $f(z) = f(\bar{z})$

Problem 1. If $h \in Harm(\mathbb{C}_+)$ and is continuous up to the boundary then

$$h(z) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{h(t)}{(x-t)^2 + y^2} dt = P_h(z)$$

Solution 1. $P_h(z)$ is harmonic in $\mathbb{C}_+$ and

$$\lim_{\substack{z\to x\in\mathbb{R} \\ z\in\mathbb{C}_+}} P_h(z) = h(x)$$

We will define $v = h - P_h$, then $v \in Harm(\mathbb{C}_+)$, $v|_{\mathbb{R}} = 0$ is bounded. By the reflection principle we can continue v to be harmonic in $\mathbb{C}$ (with $v(z) = -v(\bar{z})$) and a bounded v. Therefore by Liouville it is bounded and $v \equiv 0$.

Problem 2. Find a bijective transformation from $\mathbb{D}$ to $\mathbb{C}\backslash(0, \infty)$ such that $f(0) = 0, f'(0) > 0$

Solution 2. The first transformation is $z_1 = \frac{1+z}{1-z}$ which sends $\mathbb{D}$ to the positive half plane. In addition $z_1(0) = 1$.
The second transformation we will choose

$$z_2 = -z_2^2 = -\left(\frac{1+z}{1-z}\right)^2$$

Thus, $z_2(0) = -1$. This is still injective since we are only performing it on the half plane. Taking z_3 as follows:

$$z_3 = \frac{1-z_1^2}{4} = \frac{1}{4} - \frac{1}{4}\left(\frac{1+2z+z^2}{(1-z)^2}\right) = \frac{1}{4} - \frac{1}{4}\left(\frac{(1-z)^2}{(1-z)^2} + \frac{4z}{(1-z)^2}\right) = \frac{-z}{(1-z)^2}$$

and $\big(z_3(0)\big)' = -1$ which gives us the following function, also known as the Koebe function

$$f(z) = \frac{z}{(1-z)^2}$$

Problem 3. Find a transformation from $\mathbb{D}$ to $\mathbb{C}\backslash\big\{(\pm 1, \infty) \cup (\pm i, \infty)\big\}$. Where $f(0) = 0$ and $f'(0) > 0$

Solution 3. In the previous problem we showed that $z_1 = \frac{z}{(1+z)^2}$. Choosing $z_2 = z_1(z^4)$, meaning that $z_1 = \frac{z^4}{(1+z^4)^2}$. We will choose $z_3 = \sqrt[4]{z_2}$ using the main branch $\mathrm{Arg}(-\pi, pi]$. Thus, $z_3 = \frac{z_4}{\sqrt{1+z^4}}$ and $z_4 = \sqrt{z}z_3 = \frac{\sqrt{z}z}{\sqrt{1+z^4}}$ and by Schwarz, z_4 is the required transformation.

Problem 4. Let f be an entire function for which $f(\mathbb{R}) = \mathbb{R}$ and $f(i\mathbb{R}) = i\mathbb{R}$. Show that the function is odd. $f(z) = -f(-z)$

Solution 4. $h(z) := f(z) + f(-z)$, by symmetry $f(z) = \bar{f}(\bar{z})$ for all z. In particular for $y \in \mathbb{R}$ we have $f(iy) = \bar{f}(-iy) = -f(-iy)$. Meaning that $h(iy) = 0$ for all $y \in \mathbb{R}$ and therefore by uniqueness, $f \equiv 0$.