Complex function theory - recitation

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1 Poisson integral

Let $H : \Pi \to \mathbb{R}$ which is bounded and piecewise continuous then

$$P_H(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{i\varphi})\frac{1-|z|^2}{\left|e^{i\varphi}-z\right|^2}d\varphi$$

is the poisson integral of H and is harmonic in $\mathbb{D}$ and $P_H(z) \xrightarrow[z\to e^{i\theta}]{z\in\mathbb{D}} H(e^{i\theta})$

Problem 1 (Harnack inequality)**.** Show that if u is harmonic and positive in D then for all $z \in \mathbb{D}$

$$\frac{1-|z|}{1+|z|} \leq \frac{u(z)}{u(0)} \leq \frac{1+|z|}{1-|z|}$$

Solution 1. We will assume that $u \in C(\bar{\mathbb{D}})$, thus by Poisson, for all $z \in \mathbb{D}$ we have

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\varphi})\frac{1-|z|^2}{\left|e^{i\varphi}-z\right|^2}d\varphi$$

By the triangle inequality we have

$$1-|z| \leq \left|e^{i\varphi}-z\right| \leq 1+|z|$$

And we arrive at

$$\frac{1-|z|}{1+z} \leq \frac{1-|z|^2}{\left|e^{i\varphi}-z\right|^2} \leq \frac{1+|z|}{1-|z|}$$

Multiplying by $A = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\varphi})d\varphi = u(0)$ where the second equality comes from the average value principle.we have

$$A\cdot\frac{1-|z|}{1+z} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\varphi})\frac{1-|z|^2}{\left|e^{i\varphi}-z\right|^2}d\varphi \leq A\cdot\frac{1+|z|}{1-|z|}$$

Using the poisson integral, we have the requaired inequalities.
If $u \notin C(\bar{\mathbb{D}})$ then we will define $u_r(z) = u(r\cdot z)$ for $0 < r < 1$. $u_r \in C(\mathbb{D})$ and harmonic in $\mathbb{D}$ and therefore,

$$\frac{1-|z|}{1+|z|} \leq \frac{u_r(z)}{u(0)} \leq \frac{1+|z|}{1-|z|}$$

Thus, for all $|w| < r$ we have

$$\frac{1-\left|\frac{w}{r}\right|}{1+\left|\frac{w}{r}\right|} \leq \frac{u_r(z)}{u(0)} \leq \frac{1+\left|\frac{w}{r}\right|}{1-\left|\frac{w}{r}\right|}$$

And as $r \to 1$ we have the required inequality.

Problem 2. Show that if $u \in Harm(\mathbb{C})$ and $0 \geq u$ Then u is constant.

Solution 2. There exists an entire f such that $f = u + iv$, the function $g = e^f$ is also entire and bounded by 1, therefore g is constant $\Rightarrow$ f is constant $\Rightarrow$ u is constant.

Problem 3. Show that if f is entire and $\Re(f(z)) = \mathcal{O}\left(|z|^p\right)$ as $z \to \infty$ then f is a polynomial with a degree of at most p

Reminder 1 (Schwartz integral). If $f = u + iv$ is analytic in $|z| < r$ and continuous up to the boundary, then for all $|z| < r$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\frac{re^{i\varphi}+z}{re^{i\varphi}-z}d\varphi + iv(0)$$

Solution 3. For $|z| < r$ (we will choose r at the end)

$$\left|f(z)\right| \leq \frac{1}{2\pi}\int_0^{2\pi} Cr^p\frac{r+|z|}{r-|z|}d\varphi + \left|v(0)\right|$$

Choosing $r = 2|z|$ we ahve

$$\left|f(z)\right| \leq C \cdot 2^p|z|^p \cdot 3 + \left|v(0)\right|$$

And according to one of the corollaries of the Louiville theorem f is a polynomial with a degree of at most p.

Problem 4 (2010)**.** Le f be an analytic function in $|z| < 1$ and continuous up to the boundary. Assume that $1 < \left|f(z)\right| < M$ for $|z| = 1$ and $f(0) = 1$.

[a.]Show that there exists a $|z_0| < 1$ such that $f(z_0) = 0$. Show that the aforementioned z_0 holds $|z_0| > \frac{1}{M}$

Solution 4. 1. Assume by contradiction that $f \neq 0$ in $|z| < 1$ we will define an analytic function $g = \frac{1}{f}$, thus by the maximum principle

$$1 = \left|g(0)\right| \leq \max_{|z|=1}|g| < 1$$

Thus we have a contradiction and there exists a z_0 such that $f(z_0) = 0$.

2. Let $|z_0| < 1$ for which $f(z_0) = 0$ we will define

$$g(z) = \begin{cases} \frac{f(z)}{z-z_0} & z \neq z_0 \\ \lim_{z\to z_0}\frac{f(z)}{z-z_0} & z = z_0 \end{cases}$$

g is analytic in $|z| < 1$ and continuous up to the boundary. $g(0) = -\frac{1}{z_0}$. And if $|z| = 1$ then $\left|g(z)\right| = \frac{\left|f(z)\right|}{|z-z_0|} < \frac{M}{|z-z_0|}$ And by the maximum principle

$$\frac{1}{|z_0|} = \left|g(0)\right| \leq \max_{|z|=1}\left|g(z)\right| < \max_{|z|=1}\frac{M}{|z-z_0|}$$

And here the TA got mixed up and will upload an answer later.