Complex function theory - recitation

Kiro Avner
Arazim ©

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Theorem 1. *Let G be a good domain and f be a meromorphic function. If $f \neq 0, \infty$ on ∂G then*

$$\frac{1}{2\pi i}\int_{\partial G}\frac{f'}{f} = N_f(G) - P_f(G)$$

Including multiplicity, where N_f is the number of zeores and P_f is the number of poles.

Theorem 2 (Rouche theorem)**.** *Let G be a good domain and $f, g \in Hol(G)$ are continuous up to the boundary. If $|f-g| < |g|$ in ∂G then $N_f(G) = N_g(G)$*

Problem 1. How many solution does the equation $2z^5 + 8z - 1 = 0$ have in the annulus $a < |z| < 2$

Solution 1. Denoting $f(z) = 2z^5 + 8z - 1$ and $g(z) = 2z^5$. Then $|f-g| = |8z-1| \overset{|z|=2}{\leq} 17$ and $|g| = 2^6 > 17$. Therefore, by the Rouch theorem, f, g have an identical number of zeroes in $|z| < 2$(5 zeroes). For $|z| < 1$ we will define $h(z) = 8z$.

$$|f-h| \overset{|z|=1}{=} \left|2z^5 - 1\right| \leq 3 < 8 = |h|$$

and consequently by the Rouche theorem f has a single 0 in $|z| < 1$. In total, f has 4 seroes in the aforementioned annulus, we can easily check that f has no zeroes for $|z| = 1$

Problem 2. Prove that the equation $(z-2)^3e^z = 1$ has 3 different solutions with a multiplicity of 1 in $\{\Re z \geq 0\}$

Solution 2. We will define $f(z) = (z-2)e^z - 1$. First we will show taht ecery solutin to $f(z) = 0$ has a degree of 1. Assume that $f(z) = 0$ and therefore $f'(z) = 3(z-2)^2e^z + \overbrace{e^z(z-2)^3}^{=1}$. If $f'(z) = 0$ we put in $f(z)$ and get that $\Re z \leq 0$.
It is sufficient to hsow that the function $h(z) = (z-2)^3 - e^{-z}$ jas exactly 3 solutions in the right half plane. We will define $g(z) = (z-2)^3$

$$\left|g(z) - h(z)\right| = \left|e^{-z}\right| \leq 1 \qquad \left|g(z)\right| \geq \min\left\{(R-2)^3, 8\right\}$$

For any big enough $R \geq 4$ we have $|g-h| < |g|$ therefore by the rouche therorem we have 3 zeroes in the domain, setting $R \to \infty$ and we have the required terms.

Theorem 3 (Open mapping theorem)**.** *If f is analytic in the domain D and is not a constant then $f(D)$ is open.*

Problem 3. Let f be entire and one-to-one, show that f is linear.

Solution 3. If f is a polynomial then $f'(z) \neq 0$ for all z (from injectivity), f' is a polynomial and by the fundamental theorem, f' is constant and therfore linear.
If f is not a polynomial then f has an essential singularity at infinity and the set $G_1 = f(\{|z| > 1\}$ is dense (Cazoratti weierstrauss). By the open mapping theoremm $G_2 = f(\{|z| < q\}$ is open, therforem $G_1 \cap G_2 \neq \emptyset$ and there exist $|z_1| < 1$,$|z_2| > 1$ such that $f(z_1) = f(z_2)$.

Lemma 1 (Schwartz's lemma)**.** *If $f : D \to D$ ($D = \{|z| < 1\}$) is analytic such that $f(0) = 0$ then $\left|f(z)\right| < |z|$ and $\left|f'(0)\right| < 1$. In addition, if there exists a w such that $\left|f(w)\right| = |w|$ or $f'(0) = 1$ then there exists a $|C| = 1$ such that $f(z) = c \cdot z$*

Note 1. The function $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$. For $a \in D$ we have $\phi_a : D \to D$ and is injective, $\phi_a(a) = 0$.

Problem 4. $f(z) = \sum_{n \geq 0} a_n z^n$ is analytic in D and bounded $\left|f(z)\right| \leq M$ in D. Show that $M|a_1| \leq M^2 - |a_0|^2$

Solution 4. Assume that f is not constant. we will define $g = \frac{f}{M}$, $g : D \to D$. $g(0) = a_0$. We will define $h = \phi_{a_0/M} \circ g$. $h : D \to D$, $h(0) = 0$ and therefore by Schwarz's lemma, $\left|h'(0)\right| \leq 1$.