Complex function theory - recitation

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Problem 1. Find $\sqrt[5]{1-i}$ (Not well defined).

Solution 1. We are asked to solve $z^5 = 1-i$. We will mark $z = re^{i\theta}$

$$r^5 e^{5i\theta} = \sqrt{2}e^{-i\frac{\pi}{4}}$$

Thus, $r = 2^{1/10}$ and $5\theta_k = -\frac{\pi}{4} + 2\pi k, k \in \mathbb{Z}$. $\theta_k = -\frac{\pi}{20} + \frac{2\pi}{5}k$ where $k \in \{0, 1, \dots, 4\}$

Problem 2. Show that $z + \frac{1}{z} \in \mathbb{R}$ iff $z \in \mathbb{R}\setminus\{0\}$ or $|z| = 1$.

Solution 2. If $z \in \mathbb{R}\setminus\{0\}$ then $z + \frac{1}{z} \in \mathbb{R}$. If $|z| = 1$ there exisrs a θ such that $z = e^{i\theta}$ and thus $z + \frac{1}{z} = e^{i\theta} + e^{i\theta} = 2\Re\left(e^{i\theta}\right) = 2\cos\theta \in \mathbb{R}$.
In the other direction, we will assume that $z = re^{i\theta}$ and $z + \frac{1}{z} \in \mathbb{R}$

$$re^{i\theta} + \frac{1}{r}e^{-i\theta} = z + \frac{1}{z} \in \mathbb{R} \Leftrightarrow \Im\left(re^{i\theta} + \frac{1}{r}e^{i\theta}\right) = 0$$

$$r\sin\theta + \frac{1}{r}\sin(-\theta) = 0 \Leftrightarrow (r - \frac{1}{r}\sin\theta = 0 \Rightarrow \theta = \pi k, r = 1$$

Problem 3. For $0 < \theta < \pi$ calculate arg $\left(e^{i\bar{\theta}} + 1\right)$

Solution 3. We want to find $\arg(e^{-i\theta} + 1)$

$$e^{i\theta} + 1 = (\cos\theta + 1) - i\sin\theta = 2\cos^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\arg(e^{i\theta} + 1) = \tan^{-1}\left(-\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) = -\frac{\theta}{2}$$

And $\left|\frac{\theta}{2}\right| \leq \frac{\pi}{2}$.

Problem 4. Find the geometric location of all of the points $z \in \mathbb{C}$ such that $|z|^2 = 2\Re(z)$

Solution 4. We will define $z = x+yi$ then z follows the required equation iff $x^2+y^2 = 2x \Rightarrow (x-1)^2+y^2 = 1 \Rightarrow |z-1|^2 = 1 \Leftrightarrow |z-1| = 1$

Problem 5. Show the following identity:

$$\left|\frac{z}{|z|} - |z| \cdot w\right| = \left|\frac{w}{|w|} - |w| \cdot z\right|$$

For $z, w \in \mathbb{C}\setminus\{0\}$

Solution 5. We will square both sides:

$$\left|\frac{z}{|z|} - |z| \cdot w\right|^2 = \left|\frac{w}{|w| - |w| \cdot z}\right|^2 \Leftrightarrow (\frac{z}{|z|} - |z| \cdot w)(\frac{\bar{z}}{|z|} - |z| \cdot \bar{w}) = \left(\frac{w}{|w| - |w| \cdot z}\right)\left(\frac{\bar{w}}{|w| - |w| \cdot \bar{z}}\right)$$

After opening up the brackets we arrive at $1 - z\bar{w} - w\bar{z} + |z|^2|w|^2 = 1 - z\bar{w} - w\bar{z} + |z|^2|w|^2$ or, as they say in my home country, mashal.