Calculus 2

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1 Summation by parts

$\{a_n\}, \{b_n\}$ are sequences.

$$\sum_{k=m}^{n} a_k(b_{k+1} - b_k) = a_{n+1}b_{n+1} - a_m b_m - \sum 6n_{k=m} b_{k+1}(a_{k+1} - a_k) \tag{1}$$

$$\sum_{k=0}^{n}(a_k - b_k a_0 \sum_{k=0}^{n} b_k + \sum_{j}^{n-1}\left((a_{j+1} - a_j)\sum_{k=j+1}^{n} b_k\right) \tag{2}$$

$$\sum_{k=0}^{n}(a_k \cdot b_k a_n \cdot \sum_{k=0}^{n} b_k - \sum_{j=0}^{n-1}\left((a_{j+1} - a_j)\sum_{k=0}^{j} b_k\right) \tag{3}$$

1.1 Abel and Dirichlet

Theorem 1 (Dirichlet). *Given $\{a_n\}, \{b_n\}$ sequences of functions in the area D.*

- *The series $\left\{B_N(x) = \sum_{n=1}^{N} b_n(x)\right\}$ is bounded uniformly in D.*
- *for all x $\left\{a_n(x)\right\}$ is monotonic and in addition, $a_n(X) \xrightarrow{u} 0$*

then the series $\sum a_n(x)b_n(x)$ unifomly converges in D.

Theorem 2 (Abel). *Given $\{a_n\}, \{b_n\}$ sequences of functions in the area D.*

- *The series $\left\{B_N(x) = \sum_{n=1}^{N} b_n(x)\right\}$ uniformly converges in D. (The series$\sum_{n=1}^{\infty} B_N(x) = \sum_{n=1}^{N} b_n(x)$ uniformly converges in D).*
- *For all $x \in D$ $\left\{a_n(x)\right\}$ is monotonic and bounded uniformly*

then the series $\sum a_n(x)b_n(x)$ unifomly converges in D.

1.2 A useful Lemma

Lemma 1. *for all $x \in (0, \pi)$*

$$\left|B_n(x)\right| = \left|\sum_{n=1}^{N} \sin(nx)\right| \leq \frac{1}{\sin(\frac{x}{2})}$$

Proof. We will notice that:

$$\sin^{\alpha} \cdot \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$B_N(x) \cdot \sin\left(\frac{x}{2}\right) = \sum_{n=1}^{N} \sin(nx) \cdot \sin\left(\frac{x}{2}\right) = \frac{1}{2}\sum_{n=1}^{N}\left[\cos\left((n-\frac{1}{2})\cdot x\right) - \cos\left((n+\frac{1}{2})\cdot x\right)\right] =$$

$$\frac{1}{2}\left[\cos\left(\frac{x}{2}\right) - \cancel{\cos\left(\frac{3x}{2}\right)} + \cancel{\cos\left(\frac{3x}{2}\right)} - \cancel{\cos\left(\frac{5x}{2}\right)} + ... + \cancel{\cos\left((N-\frac{1}{2})\cdot x\right)} - \cos\left((N+\frac{1}{2})\cdot x\right)\right]$$

$$\Rightarrow \left|B_N(x)\right| \cdot \sin\frac{x}{2} \leq \left|\frac{\cos\left(\frac{x}{2}\right) - \cos\left((N+\frac{1}{2})\cdot x\right)}{2}\right| \Rightarrow \left|B_N(x)\right| \leq \frac{1}{\sin(\frac{x}{2})}$$

□

1.3 Problems

1. Find the sums of the following series:

   (a)

   $$\sum_{k=0}^{n} k \cdot 2^k$$

   We will use the third formula:

   $$\sum_{k=0}^{n} a_k \cdot b_k = n \cdot \sum_{k=0}^{n} 2^k - \sum_{j=0}^{n-1}\left((j+1)-j) \cdot \sum_{k=0}^{j} 2^k\right) =$$

   $$n \cdot (2^{n-1}-1) - \sum_{j=0}^{n-1}(2^{j+1}-1) = n \cdot 2^{n-1} = n + n - (2^{n+1} - 2 = ... = (n-1)2^{n+1} + 2$$

   (b)

   $$\sum_{k=1}^{n} k^2$$

   For the sequences $a_k = k^2$ and $b_k = k$ and using the first formula we get that

   $$\sum_{k=1}^{n} k^2 = \overbrace{k^2}^{a_k} \cdot \overbrace{((k+1)-k)}^{b_{k+1}-b_k=1} = \left((n+1)^2(n+1)-1\right) - \sum_{k=1}^{n}(k+1)\left((k+1)^2-k^2\right)$$

   $$= (n+1)^3 - 1 - \sum_{k=1}^{n}(k+1)(2k+1) = (n+1)^3 - 1 - \sum_{k=1}^{n}(2k^2+3k+1) =$$

   $$= (n+1)^3 - 1 - 2\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k - n \Rightarrow 3\sum_{k=1}^{n} k^2 = ... = n^3_{\frac{3n^2}{2}} + \frac{n}{2} \Rightarrow \sum_{k+1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

2. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges in $\setminus$ to a continuously differentiable function.

Solution: set $x \in \mathbb{R}$ and we will show that the infinite series is differentiable in at x. We will look at the section $[-A, A]$ for $A = |x| + 1$ (and, in particular $x \in [-A, A]$).
We will look at the series of derivatives:

$$u_n'(x) = \frac{(-1)^n}{x^2 + n} = (-1)^n \cdot \frac{1}{x^2 + n} = b_n(x) \cdot a_n(x)$$

we get that $B_n(x) = \sum_{n=1}^{N} (-1)^n$ is uniformly bounded by 2. Also , $\{a_n(x)\}$ is monotonic and

$$|a_n(x)| \leq \frac{1}{n} \to 0 \Rightarrow a_n(x) \xrightarrow{u} 0$$

From the Dirichlet Theorem the derivative series uniformly converges in $[-A, A]$. In addition, w ewill notice that for $x_0 = 0$the series $\sum u_n(x_0)$ converges to 0.and therefroe from a theorem on differentiation item-item there exists a function $f : [-A, A] \to \setminus$ such that

(a) $f'(x) = \sum_{n=1}^{\infty} u_n'(x)$ for all $x \in [-A, A]$

(b) f' is continuous as a uniform limit of a continuous function.

(c) $f(x) = \sum_{n=1}^{\infty} u_n(x)$ for all $x \in [-A, A]$

and in particular, the series converges to a uniformly differentiable series.

3. Prove that the series $\sum_{n=1}^{N} \frac{x \sin(nx)}{n}$ uniformly converges in the section $[\delta, \pi - \delta]$ for all $\delta \in (0, \pi)$.

Solution We will mark $a_n(x) = \frac{x}{n}, b_n(x) = \sin(nx)$ then, fro mthe previous Lemma, $\sum_{n=1}^{N} b_n(x)$ is uniformly bounded by $\frac{1}{\sin(\frac{\delta}{2})}$ conversely, $\{a_n(x)\}$ is monotonic

4. Does the following series uniformly converge in $\mathbb{R}$?

$$\sum_{n=1}^{\infty} \frac{x^2(-1)^{n-1}}{(x^2 + 1)^n}$$

Solution: We will define $b_n(x) = (-1)^{n-1}$ The series$\sum_{n=1}^{N} b_n(x)$ is uniformly bounded. We will define $a_n(x) = \frac{x^2}{(x^2+1)^n}$

5. Given $\{\lambda_n\}$ a series of monotonic rising numbers that is non-negative from a certain point. And given $x_0 \in \mathbb{R}$such that the series $\sum_{n=1}^{\infty} a_n \cdot e^{-\lambda_n x_0} < \infty$ prove that the series $\sum_{n=1}^{\infty} a_n \cdot e^{-\lambda_n x_0}$ uniformly converges in $[x_0, \infty)$

Solution We will notice that $a_n e^{-\lambda_n x} = \overbrace{\left(a_n e^{-\lambda_n x}\right)}^{b_n(x)} \overbrace{e^{-\lambda_n \underbrace{x - x_0}_{\text{Positive}}}}^{C_n(x)}$

(a) $\left\{\sum_{n=1}^{N} b_n(x)\right\}$ uniformly converges in $[x_0, \infty)$

(b) Since $\{\lambda_n\}$ is a monotic rising sequence $\Rightarrow e^{-\lambda_n}$ is a monotonic decreasing sequence $\Rightarrow C_n(x) = e^{-\lambda_n(x-x_0)}$which is a monotic decreasing sequence for all x and is bounded by 1.

Consequently, from the Abel theorem we get that the series uniformly converges in $[x_0, \infty)$

2 Power Series

Definition: A power series around $z_0 \in \mathbb{C}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n$$

$\{a_n\} \subseteq \mathbb{C}$ is a series of numbers.

Note 1.
- For $z_0 \neq 0$ we can switch the variables $w = z - z_0$ and look at a power series around 0 instead of around z_0
- If $z = z_0$ then the series $\sum_{n=0}^{\infty} a_n((z - z_0)^n = a_0$ converges
- It is east to create a series that does not converge at at points other than z_0 for example $a_k = k!$ and it is also easy to create a series that converges in all of $\mathbb{C}$ for example $a_k = \frac{1}{k!}$

2.1 Theorems

Theorem 3 (Abel)**.** *For all power series* $\sum_{k=0}^{\infty} a_n \cdot z^k$ *there exists an* $R \in [0, \infty]$ *such that for all*

- $\{|z| > R\}$ *the series diverges.*
- $\gamma < R$ *the series uniformly converges (absolutely in the section* $\{|z| \leq \gamma\}$*)*

Corollary 1. *The series converges to a continuous function in the section* $\{|z| < R\}$ R *is called the convergence radius of the series.*

Theorem 4 (Cauchy Hadamard)**.**

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Where "$\frac{1}{\infty}$" $= 0$ *and* "$\frac{1}{0}$" $= \infty$

Theorem 5 (Delambert theorem)**.** *if the limit* $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ *exists, then* $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

2.2 Known Series

Function	Series	Convergence radius
e^x	$\sum \frac{x^k}{k!}$	∞
	$\sum k! \cdot x^k$	0
$\ln(1+x)$	$\sum \frac{x^k}{k}(-1)^n$	1
$\frac{1}{1-x}$	$\sum x^k$	1
$\frac{1}{(1-x)^2}$	$\sum (k+1)x^k$	1

Function	Series	Convergence radius
$\sin x$	$\sum \frac{x^{2k+1}\cdot(-1)^k}{(2k+1)!}$	∞
$\cos x$	$\sum \frac{x^{2k}\cdot(-1)^k}{(2k)!}$	∞
$\arctan x$	$\sum \frac{(-1)^k x^{2k+1}}{2k+1}$	1
$\frac{1}{1+x^2}$	$\sum (-1)^k x^{2k}$	1

2.3 Problem

Find the convergence radius for the following series:

1.
$$\sum_{n=1}^{\infty} \frac{(z-2)^n}{n^{1/n}}$$

Solution: in this case, $z_0 = 2$ and $a_n = \frac{1}{n^{1/n}}$ Therefore:

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^{1/n}}} = \lim_{n\to\infty} \frac{1}{n^{1/n^2}} = 1 \Rightarrow R = 1$$

2.
$$\sum_{n=0}^{\infty} \frac{(n+1)z^{3n}}{2^n}$$

Solution:

- **First Way:** We will look at the series $\sum_{n=0}^{\infty} \frac{(n+1)w^n}{2^n}, w = z^3$ We will get that if R_0 is the convergence radius for this radius then our convergence radius is $R = \sqrt[3]{R_0}$
- **Second Way:**

$$a_n = \begin{cases} \frac{k+1}{2^k} & n = 3k \\ 0 & \text{else} \end{cases}$$

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{k\to\infty} \sqrt[3k]{\frac{k+1}{2^k}}$$

3.
$$\sum_{n=1}^{\infty} \frac{n^2(z+1)^n}{\pi^n + e^n}$$

Solution In this question, $a_n = \frac{n^2}{\pi^n+e^n}, z_0 = (-1)$

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \frac{1}{\pi} \frac{\sqrt[n]{n^2}}{\sqrt[n]{1 + \left(\frac{e}{\pi}\right)^n}} = ... = \frac{1}{\pi} \Rightarrow R = \pi$$

4. $\sum_{n=1}^{\infty} \frac{x^n}{n^2 \cdot 2^n}$ Find the convergence area in $\mathbb{R}$.

Solution:

$$x_0 = 0, a_n = \frac{1}{n^2 \cdot 2^n}, \frac{1}{R} = ... = \frac{1}{2} \Rightarrow R = 2$$

We will checkthe boundary points which are ± 2. It is easy to see that $\sum \frac{2^n}{n^2 \cdot 2^n} < \infty$ and therefore we have absolute convergence in $[-2, 2]$.

5.
$$\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{n}$$

Find the convergence area in $\mathbb{R}$.

Solution: $x_0 = -1, c_n = \frac{3^n+(-2)^n}{n}$ Can we do this:

$$\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{n}(x+1)^n \overset{?}{=} \sum_{n=1}^{\infty} \frac{3^n}{n}(x+1)^n + \sum_{n=1}^{\infty} \frac{(-2)^n}{n}(x+1)^n$$

The step ? is legal if both absolutely converge.

From Homework 7: Let $\sum a_n x^n, \sum b_n x^n$ be series with a convergence radius of R_1, R_2 respectively. If $R_1 \neq R_2$ then the convergence radius of the series $\sum_{n=1}^{\infty}(a_n + b_n)x^n$ is $\min\{R_1, R_2\}$.

$$\frac{1}{R_1} = \limsup_{n\to\infty} \sqrt[n]{\frac{3^n}{n}} = 3 \Rightarrow R_1 = \frac{1}{3}$$

$$\frac{1}{R_2} = \limsup_{n\to\infty} \sqrt[n]{\frac{|-2|^n}{n}} = 2 \Rightarrow R_2 = \frac{1}{2}$$

And therefore the convergence radius is $\frac{1}{3}$.

2.4 Integration and differentiation

Theorem 6. *The series $\sum a_k \cdot z^k$ and the series $\sum k \cdot a_k \cdot z^{k-1}$ have the same convergence radius.*

Corollary 2. *If the series $\sum a_k x^k$ converges in the section $(-R, R)$ (R is the convergence radius). Then for all $x \in (-R, R)$ $\sum k \cdot a_n \cdot z^{k-1}$ converges to $f'(x)$ and in particular in the section $(-R, R)$ f is differentiable in all orders.*

Corollary 3. *The Taylor expansion of f around 0 is $\sum a_k x^k$ if for all k $a_k \frac{f^{(k)}(0)}{k!}$*

Note 2. There exist series with Taylor expansions that converge but not to a function. For example:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f is differentiable ∞ times around 0 and $f^{(m)}(0) = 0$ in particular the Taylor expansion converges in all of $\mathbb{R}$ but not to a function.

2.5 Problem

Find a Taylor expansion for the function $f(x) = e^{-x^3}$ around $x_0 = 0$

2.5.1 Solution

A known Taylor expansion is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{3n}}{n!}$$

There is convergence in all of $\mathbb{R}$ to a function from the uniqueness of Taylor expansions.

2.6 Problem

Find a closed expression for the following series and write it if we have an equality.

1. $$S(x) = \sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$$

   WE will use the Taylor expansion of e^{x+2}

   $$e^{x+2} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} = \sum_{n=3}^{\infty} \frac{(x+2)^n}{n!} + \frac{(x+2)^2}{2} + x + 2 + 1 \overset{k=n-3}{=} \sum_{k=0}^{\infty} \frac{(x+2)^{k+3}}{(k+3)!} + \frac{(x+2)^2}{2} + x + 2 + 1$$

$$= (x+2)^3 \sum_{k=0}^{\infty} \frac{(x+2)^{k+3}}{(k+3)!} + \frac{(x+2)^2}{2} + x + 3 = (x+2)^3 \cdot S(x) + \frac{(x+2)^2}{2} + x + 3 = e^{x+2}$$

$$x \neq -2: \qquad S(x) = \frac{1}{(x+2)^3} \left(e^{x+2} - \frac{(x+2)^2}{2} - x - 3 \right), \qquad x = -2: \qquad S(x) = \frac{1}{3!}$$

2.

$$S(X) = \sum_{n=0}^{\infty} \frac{x^{m=1}}{2n+1}$$

$$S'(x) = \sum_{n=0}^{\infty} (x^2 n) \stackrel{|x|<1}{=} \frac{1}{1-x^2}$$

$$S(x) = \int_0^x \frac{1}{1-t^2} = ... = \ln \left(\sqrt{\frac{1+x}{1-x}} \right)$$

3.

$$S(x) \sum_{n=0}^{\infty} (n+1) x^n$$

We will notice that

$$\sum (x^n)' = \int_0^x S(x) = \sum_{n=1}^{\infty} x^{n+1} = \sum_{n=2}^{\infty} x^n = \stackrel{|x|<1}{=} \frac{x^2}{1-x} \Rightarrow S(x) = ????$$

4.

$$S(x) = \sum_{n=0}^{\infty} \frac{n^2 + !}{2^n \cdot n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{n^2+1}{n!} \cdot \left(\frac{x^2}{2} \right)^n = \sum_{n=0}^{\infty} \frac{n^2}{n!} \cdot \left(\frac{x^2}{2} \right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{x^2}{2} \right)^n = I(x) + e^{\frac{x}{2}}$$

$$I(x) = \sum_{n=0}^{\infty} \frac{n^2}{n} \cdot \left(\frac{x}{2} \right)^n \stackrel{k=(n-1)}{=} \sum_{k=0}^{\infty} \frac{(k+1)^2}{(k+1)!} \cdot \left(\frac{x}{2} \right)^{k+1} = \sum_{k=0}^{\infty} \frac{k+1}{k!} \cdot \left(\frac{x}{2} \right)^{k+1} \stackrel{m=k-1}{=}$$

$$= \sum_{m=0}^{\infty} \frac{\overbrace{m+2}^{m+1+1}}{(m+1)!} \cdot \left(\frac{x}{2} \right)^{m+2} = \sum_{m=0}^{\infty} \frac{1}{m!} \cdot \left(\frac{x}{2} \right)^{m+2} + \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \cdot \left(\frac{x}{2} \right)^{m+2} = \left(\frac{x}{2} \right)^2 \cdot e^{\frac{x}{2}}$$