Calculus 2

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1 Differentiation and Integration of sequences of functions

Theorem 1. *$f_n : [a, b] \to \mathbb{R}, f_n \in R[a, b], f : [a, b] \to \mathbb{R}$ such that $f_n \xrightarrow{u} f$ then:*

1. *$f \in R[a, b]$*
2. *$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$*

If the integral is indefinite then even uniform convergence will not help.For example:

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in [0, n] \\ 0 & else \end{cases}$$

Theorem 2 (Majorants)**.** *$F_n[a, w] \to \mathbb{R}$ for all $w > b$ $f_n \xrightarrow{u} f$ in $[a, b]$ and there exists a $\varphi : [a, w] \to \mathbb{R}$ such that $|f_n| \leq \varphi$ and $|f| \leq \varphi$ and $\int_a^w \varphi$ converges then $\int_a^w f$ converges too. Even more so:*

$$\int_a^w f = \int_a^w \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^w f_n \leq \int_a^w \varphi$$

1.1 Problem

$f_n : [0, 1] \to \mathbb{R}$ $f_n(x) = x \cdot n^\alpha \cdot e^{-nx}$ For what values of α does $\{f_n\}$uniformly converge?

1. It is easy to see that $f_n \xrightarrow{p} 0$ since a polynomial growth rate is always smaller than an exponential one. In order to find the maximum we will use differentiation:

$$f_n'(x)n^\alpha[e^{-nx} + x \cdot (-n)e^{-nx}] = 0 \Leftrightarrow x = \frac{1}{n}$$

$$f_n(0) = 0, f_n(1) = n^\alpha e^{-n}, f\left(\frac{1}{n}\right) = \frac{s^{\alpha-1}}{e}$$

$$\sup_{[0,1]}|f_n - f| = \max\left\{n^\alpha \cdot e^{-n}, \frac{n^{\alpha-n}}{e}\right\} \overset{\text{Large enough n}}{\leq} \frac{n^{\alpha-1}}{e}$$

First case: $\alpha < 1$ then $\sup_{[0,1]}|f_n| \to 0$ and therefore w e have convergence/

Second case: $\alpha \geq 1$ then $\sup_{[0,1]}|f_n| \geq \frac{1}{e} > 0$ and in particular we do not have convergence $(x_n\frac{1}{n} \Rightarrow \left|f_n(x_n) - f(x_n)\right| \geq \frac{1}{e}$ for all n).

2. For what values of α

$$\int_0^1 \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_0^1 f_n$$

We will notice from (a) that $\int_b^a \lim_{n\to\infty} f_n = 0$

First Case $\alpha < 1$ from a theorem on uniform convergence wecan switch, in particular $\star$ agrees with it.

Second case $\alpha > 1$

$$\int_0^1 x \cdot n^\alpha \cdot e^{-nx} dx = ... = \overbrace{-\frac{n^{\alpha-1}}{e^n} - \frac{n^{\alpha-1}}{e^n}}^{\to 0} + n^{\alpha-1}$$

for the third item $n^{\alpha-1} \to 0$ iff $\alpha \leq 1$ therefore what we wanted to prove is true also for $\alpha > 1$

This problem is good example for a case when

$$f_n \overset{u}{\not\to} f \text{ and } \int_a^b f_n \to \int 6b_a f$$

2 Series of functions

Let $u_n : [a,b] \to \mathbb{R}$ ans we will mark $s_n[a,b] \to \mathbb{R}$, $S_n(x) \sum_{k=1}^n u_k(x)$ then $s_n : [a,b] \to \mathbb{R}$ we will say that the series $\sum u_n$ converges pointwise if $\{s_n\}$ pointwise converges.

2.1 Theorems

Theorem 3 (Dini)**.** *$u_n : [a,b] \to \mathbb{R}$ be continuous and non-negative. and we will assume that $f \overset{p}{\to} \sum_{n=1}^\infty u_n$. f is continuous then the convergence is uniform.*

Theorem 4 (Weierstrauss M-test)**.** *if we have a $u_n : I \to \mathbb{R}$ I is a segment and we will assume that we have a sequence of numbers M_n such that*

1. $$M_n \geq \left|u_n(x)\right| \text{ For all } x \in I$$

2. $$\sum_{n=1}^\infty M_n < \infty \text{ Then } \left\{\sum_{n=1}^N u_n\right\} \text{ Uniformly converges}$$

From the limit switching theorem We get that if $u_n : I \to \mathbb{R}$ is a series of functions that uniformly converges in I. If $a \in \bar{I}$ and we will assume that for all n, $\lim_{x\to a} u_n(x) = c_n$ and $\sum c_n < \infty$ then

$$\lim_{x\to a} \sum_{n=1}^\infty u_n(x) = \sum_{n=1}^\infty c_n$$

Corollary 1. *if u_n are continuous then $\sum u_n$ is continuous.*

Theorem 5 (Limit Switching)**.** *if we have a $g, f_n : [1,b] \to \mathbb{R}$ such that $f \in C^{-1}$ we will also assume that*

1. *there exists a $x_0 \in [a,b]$ such that $\{f_n(x_0)\}$ converges.*
2. $f'_n \overset{u}{\to} g$

Then there exists $f : [a,b] \to \mathbb{R}$ such that $f_n \overset{u}{\to} f$ and $f' = g$ or in other words, $f'_n \overset{u}{\to} f'$

2.1.1 Problems

- Prove that

$$\lim_{x\to\infty}\sum_{n=1}^{\infty}\frac{x^2}{1+n^2x^2}=\frac{1}{\pi}$$

Solution: We will notice that for all $x \in \mathbb{R}$

$$u_n(x)=\frac{x^2}{1+x^2\cdot n^2}=\frac{1}{\frac{1}{x^2}+n^2}<\frac{1}{n^2}$$

If the inequality above would only happen for $x > x_0$ it would still be OK, for this question, it would be sufficient that we would have uniform convergence in the segment $[x_0, \infty)$ for $x \in \mathbb{R}$.

We will choose $M_n = \frac{1}{n}$ then from the M-test $\sum u_n$ uniformly converges. From the uniform convergence we can use the lmit switching theorem since

  - $\sum u_n$ uniformly converges in $\mathbb{R}$.
  - $\frac{1}{n^2} = c_n = \lim_{x\to\infty} u_n(x)$ and as we know, $\infty > \frac{1}{n^2}$

Therefore from the limit switching theorem we can do:

$$\lim_{x\to\infty}\sum_{n=1}^{\infty}u_n(x)=\sum_{n=1}^{\infty}\lim_{x\to\infty}u_n(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}\overset{\text{From Calc 1}}{=}\frac{\pi^2}{6}$$

- If $R > 1$ does the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{x+n}$ uniformly converge in $[1, R]$ (Does the series converge in every subsection of $[1, \infty)$?

Notice that for all n

$$u_n(x)=\frac{(-1)^n}{x+n}\in C^1$$

and

$$u_n'(x)-\frac{(-1)^{n+1}}{(x+n)^2}\Rightarrow\left|u_n'(x)\right|\le\frac{1}{(n+x)^2}<\frac{1}{n^2}$$

from the M-test we get that the derivative uniformly converges in $[1, R]$ for all $R > 0$ and from Leibniz we get that the series $\sum u_n(x)$ converges pointwise.

Note 1. It is sufficient to check if $\sum u_n$ is a convergent series.
From the derivative theorem we get that the series $\sum u_n$ uniformly converges in $[1, R]$

Leibniz:

$$a_n \ge 0 \text{ Monotonic decreasing } \Rightarrow \sum(-1)^n a_n \text{ Converges.}$$

Theorem 6. *Let $g, f_n : [a,b] \to \mathbb{R}$ be functions so that $f_n \in C^1$ and we will also assume that*

1. *There exists an $x_0 \in [a,b]$ so that $\{f_n(x_0)\}$ converges/*
2. *$f_n' \to g$*

then there exists an $f : [a,b] \to \mathbb{R}$ so that $f_n \overset{u}{\to} g, f' = g$ or in other words $f_n' \overset{u}{\to} f'$.

> In another way, $g_n = f_n'$, g_n is continuous $\Rightarrow g_n \in R[a,b]$. $g_n \overset{u}{\to} g$ and from (2) $\Rightarrow g \in R[a,b]$. we will define $f(x) = \int_a^b g(x) \alpha x (+c)$

2.2 Problem

1. Prove that for all $m, n \geq 0$

$$\int_0^1 x^n \cdot \ln^m x dx = \frac{(-1)^m \cdot m!}{(n+1)^{m+1}}$$

2. Prove that

$$\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n}$$

3. Prove that the following function is a continuous function

$$\sum_{n=2}^{\infty} \ln\left(1 + \frac{x^2}{n^2 \ln^2 n}\right)$$

4. Prove that the function $\sum_{n=1}^{\infty} n^2 x^2 e^{-n^2 x}$ uniformly converges in $[0, \infty]$

Hint: notice that

$$x^{-x} = e^{-x \ln x} = \sum \frac{(-1)^n}{n!} \cdot x^n \ln^m x$$

1. $$I_{\min} \overset{\text{Int' by parts}}{=} \frac{x^{n+1}}{n+1} \cdot \ln^m x|_0^1 - \frac{m}{n+1} \int_0^1 x^n \ln^{m-1} x dx = \frac{-m}{n+1} \cdot I_{m-1}, n$$

   In total:

$$I_{m,n} = \frac{(-1)^m \cdot m!}{(n+1)^m} \cdot I_{0,n}, \qquad I_{0,n} = \frac{1}{n+1}$$

2. We will define the sequence of functions

$$u_n(x) = \frac{(-1)^n}{n!} x^n \ln^n x \in C^1[0,1]$$

   . We will differentiate $x^n \ln^n x$ and find $\max_{[0,1]} x^n \ln^n x$ (H.W. for all n the maximum is at the point $\frac{1}{e}$).

Differentiation idea: if $g_n = f_n - f$, g_n is diferentiable, then:

$$\max[a,b]|g_n| \leq \max\left\{\left|g_n(a)\right|, \left|g_n(b)\right|, \max_{x \text{ is an extremum}}\left|g_n(x)\right|\right\}$$

The goal: to find M_n in order to use in the M-test.
We will find $\max_{[a,b]}\left|u_n(x)\right|$ using the same trick and mark it with M_N¿

For every n we will define $M_n = \frac{\frac{1}{e^n}}{n!} = \frac{1}{ne^n}$ and because $\infty > \sum M_n$ we get from the M-test that the series uniformly converges. We will use the theorem on integration in the compact space $[0,1]$. $u_n \in C^1$ and in particular $u_n \in R[0,1]$ and therefore $\sum u_n \in R[0,1]$ and

$$\int_0^1 \sum u_n = \sum \int_0^1 u_n$$

and from the first section we have for all n: $I_{0,n} = \frac{(-1)^n \cdot n!}{(n+1)^{n-1}}$

$$\int_0^1 u_n(x) = \int \frac{(-1)^n}{n!} \cdot x^n \ln^n x dx = \frac{(-1)^n}{n!} \cdot I_{n,n} = \frac{(-1)^n}{n!} \cdot \frac{(-1)^n \cdot n!}{(n+1)^{n+1}} = \frac{1}{(n+1)^{n+1}}$$

From $\star$ we have

$$\int_0^1 x^{-x} dx \overset{\text{Hint}}{=} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \ln^n x dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n \ln^n x dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} n^{-n} \quad \square$$

3. **Idea:** It is sufficient to show that the series uniformly converges since we can switch limits ($\ln\left(1 + \frac{x^2}{n^2 \ln^2 n}\right)$ is continuous). However there is no uniform convergence. we will take $x_n = n \ln n$ and we will get $u_n(x_n) = \ln^2$ But continuity is <u>local</u> Then it is sufficient to prove uniform continuity in a compact space.

   We will set $\alpha > 0$ and look at the segment $[-\alpha, \alpha]$. we will mark $u_n(x) = \ln\left(1 + \frac{x^2}{n^2 \ln^2 n}\right)$

$$\left|u_n(x)\right| = \ln\left(1 + \frac{x^2}{n^2 \ln^2 n}\right) \overset{\ln(1+x) \leq x}{\leq} 1 + \frac{x^2}{n^2 \ln^2 n} \overset{x \in [-\alpha,\alpha]}{\leq} \frac{\alpha^2}{n^2 \ln^2 n}$$

   We will mark $M_n(\alpha) = \frac{\alpha^2}{n^2 \ln^2 n}$

$$\forall n \left|u_n(x)\right| \leq M_n(\alpha) \forall x \in [-\alpha, \alpha] \qquad \forall \alpha \in \sum_{n=1}^{\infty} M_n(\alpha) < \infty$$

   And from the M-test we will get that $\{\sum u_n\}$ uniformly converges in $[-\alpha, \alpha]$
   Let $x_0 \in \mathbb{R}$ and we will set $\alpha = |x_0| + 1$ and in particular $x_0 \in [-\alpha, \alpha]$ and now $\sum u_n$ is uniformly convergent in hte segment $[-\alpha, \alpha]$and in particular the limit function is continious(since u_n is continuous for all n.we will get that $\sum u_n(x)$ is continuous at the point x_0 since x_0 is a general point we have that $\sum u_n(x)$ is continuous.

4. **We will mark** $u_n(x) = n^2 x^2 e^{-n^2 x}$ and therefore:

$$u_n'(x) = 2xn^2 e^{-n^2 x} - n^4 x^2 e^{-n^2 x} = e^{-n^2 x} \cdot xn^2(2 - n^2 x) = 0 \Leftrightarrow x = 0$$

$$x_n = \frac{2}{n^2} \qquad \left|u_n(x)\right| \leq \max\left\{\left|u_n(0)\right|, \left|u_n(x_n)\right|\right\}$$

Note: If $x > x_n$then $u_n(x) < 0$ ans then u_n decreases in the section (x_n, ∞) and in particular

$$\lim_{x\to\infty} u_n(x) \le u_n(x_n)$$

We will choose $M_n := e^{-2} \cdot \frac{4}{n^2}$ then for all n $\max_{[0,\infty]} \left|u_n(x)\right| \le M_n$ and of course, $\sum M_n < \infty$, from the M-test the series convegres in $(0, \infty)$

When We are asked to prove a local property (differentiable/continuous, in $R[a, b]$...) we will show uniform continuity in the segemnt $[-\alpha, \alpha]$ for all α and get the required property.

3 Infinite Products

Definition 1. Let $\{a_n\}$ be a series of numbers and we will define $P_n = \prod_{k=1}^n a_k$ We will say that $\prod_{k=1}^\infty a_k$ converges if the sequence $\{p_n\}$ is a convergent sequence to a value $P \ne 0$. If $P \to 0$ then we will say that the product diverges to 0.

Claim 1.
1. $\prod_{k=1}^n a_k$ converges $\Rightarrow a_n \to 1$
2. if $a_n > 0$ then $\prod_{n=1}^\infty a_n$ converges $\Leftrightarrow \infty > \sum \ln(a_n)$
3. If $a_n \ge 0$ then the product $\prod_{n=1}^\infty a_n$ converges iff the series $\sum(a_n - 1)$ converges.

Proof.
1. We will notice that -
$$a_n = \frac{p_n}{p_{n-1}} \to \frac{p}{p} = 1$$
2. And since
$$\ln\left(\prod_{n=1}^\infty a_n\right) = \sum_{k=1}^n \ln(a_k)$$
$$\prod_{n=1}^\infty a_n = e^{\ln\left(\prod_{n=1}^\infty a_n\right)} = e^{\sum_{k=1}^n \ln a_k}$$
since $e^x \cdot x$ is a monotonic injective function...
3. We will mark $c_n = a_n - 1$ $a_n = 1 + c_n$
$a_n > 0$ from (2)
$$\prod a_n \text{ is convergent } \Leftrightarrow \sum \ln(a_n) < \infty$$
$$\ln(a_n) = \ln(1 + c_n) \overset{\substack{x\ge 0\\ \ln(1+x)\le x}}{\le} C_n$$
If $a_n \ge 1$ then we have shown here that $\sum \ln(a_n)$ converges if the series $\sum c_n$ converges (from the comparison test).
The series $\sum \ln(a_n)$ converges $\Leftrightarrow \prod a_n$ converges.
We have shown that for $a_n \ge 1$ we have $\sum(a_k - 1) < \infty \Rightarrow \prod a_n < \infty$

for all $c_1, c_2, ..., c_n \in \mathbb{R}$ positive numbers,

$$1 + \sum_{k=1}^n c_k \le \prod_{k=1}^n (c_k + 1)$$

Hint: proof by induction. We will define as before, $c_k = a_k - 1$

$$1 + \sum_{k=1}^{n} c_k \leq \prod_{k=1}^{n} (c_k + 1) = \prod_{k=1}^{n} a_k$$

We will get that $\prod_{k=1}^{\infty} a_k$ converges $\Rightarrow$ the series $\sum (a_n - 1)$ converges.

$\square$

A useful inequality: from the Taylor expansion of e^x we can get the for all $x \geq 0$ $e^x \geq x + 1$ and therefore

$$\prod_{n+1}^{N} a_n = \prod_{n=1}^{N} (1 + c_n) \leq \prod_{n=1}^{N} e^{c_n} = e^{\sum_{n=1}^{N} c_n}$$

and therefore if the series converges then the product converges.

3.1 Problem

Calculate $\prod_{n=1}^{\infty} \cos\left(\frac{\alpha}{2^n}\right)$ for $\alpha \neq 0$ **Reminder:** $\sin x = 2 \sin(\frac{x}{2}) \cos(\frac{x}{2})$

$$\sin \alpha = 2 \cos(\frac{\alpha}{2}) \cdot \sin(\frac{\alpha}{2}) = 4 \cos(\frac{\alpha}{2}) \cdot \cos(\frac{\alpha}{4}) \sin(\frac{\alpha}{4}) = ... = \left[2^n \prod_{k=1}^{n} \cos\left(\frac{\alpha}{2^k}\right)\right] \sin(\frac{\alpha}{2^n}) =$$

$$\overbrace{\prod_{k=1}^{n} \cos(\frac{\alpha}{2^n})}^{p_n} \cdot \frac{\sin(\frac{\alpha}{2^n})}{\frac{\alpha}{2^n}} \cdot \alpha \Rightarrow P_n = \frac{\sin \alpha \cdot (\frac{1}{\alpha})}{\frac{\sin(\frac{\alpha}{2})}{\frac{\alpha}{2^n}}} \xrightarrow[n \to \infty]{} \frac{sin\alpha}{\alpha}$$

3.2 Questions from tests

1. Prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + \frac{x^2}{n})^n} \rightarrow_{n \to \infty} \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$f_n(x) = \frac{1}{(1 + \frac{x^2}{n})^n} \qquad f_n(x) \xrightarrow{p} e^{-x^2}$$

we will want to show that $f_n \xrightarrow{u} e^{-x^2}$ + find a Majorant.

<u>H.W.:</u> $f_n \searrow f$ because $\left(1 + \frac{1}{n}\right)^n \nearrow e$ and because

$$\left[\left(1 + \frac{x^2}{n}\right)^n\right]^{\frac{1}{n+1}} \leq 1 + \frac{x^2}{n+1}$$

Now from Dini: $f_n \in c^1$, and $\{f_n\}$ is a monotonic sequence. f the limit function is also continuous from Dini's theorem $f_n \overset{u}{\to} f$

On the other hand, because $\{f_n\}$ is a monotic sequence and in particular $|f| \leq f_1, \qquad |f_n| \leq f_1$ in order to finish the proof we only need to show that $\int_{-\infty}^{\infty} f_1$ converges.

$$\int_{-\infty}^{\infty} f_1 = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = 2\int_0^{\infty} \frac{1}{1+x^2} dx = 2 \lim_{R\to\infty} \arctan|_0^R = \pi$$

and in particular we have found a majorant.

2. If we have

$$f_n : [0,1] \to \mathbb{R} \in C^1[0,1] \qquad \forall n \in \mathbb{N} \sup_{x\in[0,1]} \left|f_n'(x)\right| \leq 1 \qquad f_n \overset{p}{\to} f$$

Prove that $f_n \overset{u}{\to} f$

Solution: From the homework. we know that f is a Lipschitz function (f_n is Lipschitz with a constant of c).

Let $\epsilon > 0$. we will show that there exists an N such that for all $n > N$ we have $\sup_{[0,1]}|f_n - f| < \epsilon$. For $\delta = \frac{\epsilon}{6}$ for all $x, y \in [0,1] \qquad |x-y| = \delta$

$$\left|f(x) - f(y)\right| \leq |x-y| < \frac{\epsilon}{6}$$

$$\left|f_n(x) - f_n(y)\right| \leq |x-y| < \frac{\epsilon}{6}$$

> Compact: For each sequence of sets $I_n := (x_n - \delta_n, x_n + \delta_n)$ if $K \subseteq \cup I_n$ then there exists an $\{n_j\}_{j=1}^N$ such that $K \subseteq \cup_{j=1}^N I_{n_j}$

$$I_x := (x - \frac{\epsilon}{12}, x + \frac{3}{12}) \qquad \cup_{n=1}^N I_x \supseteq [0,1]$$

$$\left|f_n(x) - f(x)\right| \overset{x\in I_{x_j}}{\leq} \overbrace{\left|f_n(x) - f_n(x_j)\right|}^{\frac{\epsilon}{6}} + \overbrace{\left|f(x_j) - f(x)\right|}^{\frac{\epsilon}{6}} + \overbrace{\left|f_n(x_j) - f(x_j)\right|}^{\frac{\epsilon}{6}}$$

Formally:

$$x_j = j - \frac{\epsilon}{6} \qquad \left|x_j - x_{j+1}\right| = \frac{\epsilon}{6}$$

$$\forall x \in [0,1]\, \exists j \left|x - x_j\right| < \frac{\epsilon}{6}$$

$N = \frac{10}{\epsilon}$ then $x_N = \frac{10}{\epsilon} \cdot \frac{\epsilon}{6} > 1$ and in particular a finite number of points such that the sections cover $[0,1]$.

For all $1 \leq j \leq N$ from point wise convergence there exists an N_j such that for all $n > N_j$, $\left|f_n(x_j) - f(x_j)\right| < \frac{\epsilon}{3}$ we will choose $M = \max_{1\leq j\leq N} N_j < \infty$ for all $n > M$ we have that for any $x \in [0,1]$

$$\left|f_n(x) - f(x)\right| \leq \left|f_n(x) - \overset{x\in I_{x_j}}{f(x)}\right| \leq \left|f_n(x) - f_n(x_j)\right| + \left|f(x_j) - f(x)\right| + \left|f_n(x_j) - f(x_j)\right| = I_1 + I_2 + I_3$$