Calculus 2 - Sequences of functions

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1 Pointwise Convergence

$f_n : I \to \mathbb{R}$ we will say that f_n converges pointwise to f at point x if $\{f_n(x)\} \to f(x)$ we can say that f_n converges pointwise to f in the segment I if for all $x \in I$ $f_n(x) \to f(x)$

1.1 examples

1.

$$f_n : \mathbb{R} \to \mathbb{R} \; f_n(x) = \frac{x}{n}$$

We will fix x and then we have that $f_n(x) \to 0$ which means that for all x we have point wise convergence to 0.

$$f_n(x) = x^n \; f_n : \mathbb{R} \to \mathbb{R} \; f_n \to f(0,1] \; f(x) = \begin{cases} 0 & |x| < 1 \\ 1 & x = 1 \end{cases}$$

2.

$$f_n : \mathbb{R} \to \mathbb{R} \; f_n(x) = (2\cos x)^{-n} = \frac{1}{2\cos x)^n}$$

And then from the example $f_n(x) \to 0$ $\frac{1}{2\cos x} = 1$ then $f_n(x) \to 1$ for $x \in \{\pm\frac{\pi}{3} + 2\pi k, k \in \mathbb{Z}\}$ Here, we saw that at some point we had that $f_n(x) = 1$ and we have a function of the type t^n therefore there is a point where x converges to 1. If we look at the function of $\cos x$ we see that it equals to $\frac{1}{2}$ at $\frac{\pi}{3}$, we ill proceed to divide the function into sections where it converges. For all of the following, $f_n(x) \to 0$:

- $x \in \{(-\frac{\pi}{3}, \frac{\pi}{3}) + 2\pi k, k \in \mathbb{Z}\}$
- $x \in \{(\frac{2\pi}{3}, \frac{4\pi}{3}) + 2\pi k, k \in \mathbb{Z}\}$
- $x \in \{(-\frac{4\pi}{3}, \frac{2\pi}{3} + 2\pi k, k \in \mathbb{Z}\}$

3.

$$f_n : [0,1] \to \mathbb{R} \qquad f_n(x) = \begin{cases} x & x < \frac{1}{n} \\ \frac{2}{n} - x & \frac{1}{n} < x < \frac{2}{n} \\ 0 & \text{else} \end{cases}$$

there exists an N such that for all $n > N$ $x > \frac{2}{n} \Rightarrow f_n(x) = 0$ therefore for all $x \in (0,1]$ $x = 0 :$ $f_n(x) = 0$ and in particular$f_n \overset{p}{\to} 0$ we will check uniform convergence:

$$\sup_{[0,1]} |f_n - f| = \sup_{[0,1]} |f_n| = \frac{1}{n} \to_{n\to\infty} 0$$

which means that $f_n \rightrightarrows 0$

4. $f_n : [0,1] \to \mathbb{R}$ in a similar fashion to the previous example, we have pointwise convergence to the function 0. However $\sup_{[0,1]}|f_n - f| = 1 \not\to$ ans in particular we do not have uniform convergence.

In a wider sense, then we want to negate uniform convergence we will find a sequence of points $\{x_k\}_{k=1}^{\infty}$ and a sub sequence$\{n_k\}$ such that $\left|f_{n_k}(x_k) - f(x_k)\right| < \epsilon_0$ for all k ($\epsilon_0 > 0$ is a constant, the same one for all k) for instance, in example 2 $n_k = k, x_k = \frac{1}{k}$

2 Problems

Check the convergence of the following sequences of functions:

1. $f_n(x) = x^n - x^{n+1}$ in the section $[0,1]$
   We have pointwise convergence to $f = 0$.

$$g_n = f_n = x^n - x^{n+1} g_n'(x) = nx^{n-1} - (n+1)x^n = x^{n-1}(n - (n+1)x) = 0$$

   Suspicious extremum points, $x = 0, x_n = \frac{n}{n+1} \in [0,1]$

$$\sup_{[0,1]}|f_n - f| \le \max\{\left|f_n(0)\right|, \left|f_n(x_n)\right|, \left|f_n(1)\right|\} = \max\{0, \left|f_n(x_n)\right|\} = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) \to_{n\to\infty} 0$$

   Note: If $f_n(a,b) \to \mathbb{R}$ and $\lim_{x\to a}\left|f_n(x) - f(x)\right|$ or $\lim_{x\to b}$... do not exist, in this case we must check that: $\limsup_{\substack{x\to a\\x\to b}}\left|f_n(x) - f(x)\right| \to_{n\to\infty} 0$

2.

$$f_n : (0,1) \to \mathbb{R}\ f_n(x) = \frac{x^2}{x^2 + (nx-1)^2}$$

   We will notice that $g_n = f_n - f$ is differentiable and we would use that in order to find $\sup_{(0,1)}|g_n|$

$$g_n'(x) = f_n'(x) = ... = \frac{2x(1-nx)}{x^2 + (nx-1)^2)^2} = 0 \Leftrightarrow x = 0, x_n = \frac{1}{n}$$

$$\sup_{(0,1)}|f_n - f| = \sup|f_n| \le \max\{\left|f_n(\frac{1}{n})\right|, L - 0, L_1\}$$

$$L_{0_n} = \limsup_{x\to 0}\left|f_n(x)\right|\ L_{1_n} = \limsup_{x\to 1}\left|f_n(x)\right|$$

$$L_{0_n} = \limsup_{x\to 0}\left|f_n(x)\right| = \limsup_{x\to 0}\frac{x^2}{x^2 + (nx-1)^2} = 0$$

$$L_{1_n} = ... \to 0$$

$$f_n(\frac{1}{n}) = \frac{\frac{1}{n^2}}{\frac{1}{n^2} + (n \cdot \frac{1}{n} - 1)^2} = 1$$

   For the sequence $x_n = \frac{1}{n}$ we will get that

$$\sup_{(0,1)}\left|f_n(x) - f(x)\right| \ge \left|f_n(x_n) - f(x_n)\right| = 1$$

   and in particular we do not have uniform continuity.

3 Claim

$A = \{x_1, ..., x_m\} \subset \mathbb{R}$ is a finite set then $f_n \overset{u}{\to} f \Leftrightarrow f_n \overset{p}{\to} f$
$\Leftarrow$ Uniform continuity is always pointwise convergent.
$\Rightarrow$ for all $1 \leq j \leq m$ $f_n(x_j) \to f(x_j)$ from pointwise convergence, if we take $\epsilon > 0$ for each j there exists an N_j such that for all $n > N_j$ $\epsilon > \left|f_n(x_j) - f(x_j)\right|$ Now we will choose $N_0 = \max_{1\leq j\leq m} N_j$, then for all $n > N_0$

$$\sup_{x\in A}\left|f_n(x) - f(x)\right| = \max_{1\leq j\leq m}\left|f_n(x_j) - f(x_j)\right| < \epsilon$$

where the last inequality comes from the pointwise convergence and therefore we have uniform convergence.

A finite number of points does not affect uniform convergence (if there is pointwise convergence).

4 Definition

if we have a set of functions $\mathcal{F}$ which are all defined on the set A ($\forall f \in \mathcal{F}, f : A \to \mathbb{R}$) we can say that $\mathcal{F}$ is uniformly bounded if there exists an M such that $\sup_{\substack{a\in A\\ f\in\mathcal{F}}}\left|f(a)\right| \leq M$ or in other words,

$$\forall a \in A, \forall f \in \mathcal{F}. \left|f(a)\right| \leq M$$

4.1 Problems

Given $f_n : A \to \mathbb{R}$, $A \subset \mathbb{R}$ is a segment such that $f_n \overset{u}{\to} f$.

1. Prove that if $\{f_n\}$ is bounded uniformly by M then f is bounded by M

   Proof we will show that for all $\epsilon > 0, \sup_A\left|f(X)\right| < M + \epsilon$ and therefore f is bounded by M. We will set $\epsilon > 0$ there exists an N such that for all $n > N$ $\sup_A|f_n - f|$ from uniform continuity and in particular for all $x \in A$

   $$\left|f(x)\right| \leq \overset{<\epsilon}{\left|f(x) - f_n(x)\right|} + \overset{<M}{f_n(x)} < M + \epsilon$$

   and because this is true for all $x \in A$ we ahve $\sup_A|f| < M + \epsilon$

2. Prove that if $\sup_A|f| \leq M_0$ then there exists an N such that $\mathcal{F} = \{f_n\}_{n=N}^{\infty}$ is uniformly bounded (is uniform convergence necessary here?)

   How should we approach this problem?

   $$\left|f_n(x)\right| \leq \left|f_n(x) - f(x)\right| + \left|f(x)\right| < M_0 + 1$$

   for $\epsilon > 1$ there exists an N such that for all $n \geq N$ $\sup_A|f_n - f| < 1$ and therefore for all $x \in A$ we have that in particular $\sup_A|f_n| < M_0 + 1$ for all $n \geq N$ as in $\mathcal{F} = \{f_n\}_{n=N}^{\infty}$ is uniformly bounded by $M_0 + 1$.

3. Prove that if f_n is bounded (each one separately) then $\{f_n\}$ is uniformly bounded.

Proof we will show that the function is bounded and use 2.
For $\epsilon = 1$ there exists an N_0 such that for all $n \geq N_0$: $\sup_A |f_n - f| < 1$ and in particular

$$\sup_A \left|f_n(x)\right| \leq \sup_A \left| f(x) - \overset{<0}{f_0(x)} \right| + \sup_A \left|f_{N_0}\right| < M_{N_0} + 1$$

we will learn that f and now from 2 there exists an N such that $\mathcal{F} = \{f_n\}_{n=N}^{\infty}$ is uniformly bounded by $M_{N_0} + 2$ and we will choose

$$M = \max\{M_{N_0} + 2, \max_{1 \leq k \leq N} M_k + 2\}$$

then for all n if $n \leq N$ then $\sup_A |f_n| \leq M_n \leq M$ if $n > N$ then $f_n \in \mathcal{F}$ and in particular $\sup_A |f_n| < M_{N_0} + 2 \leq M$ in total we get uniform boundedness.

5 Claim

If $f_n \overset{u}{\to} f$ $g_n \overset{u}{\to} g$ then $f_n + g_n \overset{u}{\to} f + g$

proof if $\epsilon > 0$ there exists an N such that for all $n > N$ $\sup_A |f_n - f| < \frac{\epsilon}{2}$ and $\sup_A |g_n - g| < \frac{\epsilon}{2}$ therefore for all $n > N$:

$$\sup_A \left|(f_n + g_n) - (f + g)\right| \leq \sup_A |f_n - f| + \sup_A |g_n - g| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

therefore $\sup_A \left|(f_n + g_n) - (f + g)\right| \to_{n \to \infty} 0$ Something to think about what happens when $f_n \cdot g_n \to f \cdot g$ do we need more information?

5.1 example

if we have $a > 0$ and $f = f_0$ which is Riemann integrable in the section $[0, a]$ we will mark the sequence of functions

$$f_n(x) = \int_0^x f_{n-1}(t) dt$$

Show that $\{f_n\}$ converges uniformly.

f is Riemann integrable which implies boundedness. We will mark this boundary with M.

$$\left|f_1(x)\right| \leq \int_0^x \left|f_0(t)\right| dt < M \cdot x$$

$$\left|f_2(x)\right| \leq \int_0^x \left|f_1(t)\right| dt < \int_0^x M t dt = \frac{M \cdot x^2}{2}$$

by induction, we can show that $\left|f_n(x)\right| \leq M \cdot \frac{x^n}{n!}$ and from this we get that $f_n \overset{p}{\to} 0$ because for all $x \in \mathbb{R}$we have that $\frac{x^n}{n!} \to 0$ and $f_n \rightrightarrows 0$ because

$$\sup_{[0,a]} |f_n - \mathcal{F}| = \sup_{[0,a]} |f_n| \leq M \frac{a^n}{n!} \to 0$$

and by definition we get uniform continuity.

Note that if we take $f_n : \mathbb{R}_+ \to \mathbb{R}$ then $f_n \overset{p}{\to} 0$, what about uniform convergence?

6 Claim

if we take $f_n : [a, b] \to \mathbb{R}$ such that $f_n \overset{p}{\to} f$ in $[a, b]$ and $f_n \overset{u}{\to} f$ in (a, b) then $f_n \rightrightarrows f$ in $[a, b]$

Proof We will set $\epsilon > 0$.

1. $f_n \overset{u}{\to} f$ in $(a, b) \Rightarrow$ there exists N_1 such that for all $n > N_1$ $\sup_{[a,b]}|f_n - f| < \epsilon$
2. $f_n(b) \overset{p}{\to} f(b)$ therefore there exists N_2 such that for all $n > N_2$ $\left|f_n(b) - f(b)\right| < \epsilon$
3. The same for a, N_3

We will choose $N = \max\{N_1, N_2, N_3\}$ then for all $n > N$:

$$\sup_{[a,b]}|f_n - f| = \max\{\left|f_n(a) - f(a)\right|, \left|f_n(b) - f(b)\right|, \sup_{[a,b]}\left|f_n(x) - f(x)\right|\} < \epsilon$$

6.1 Example

$g \in C([0, 1]), g(1) = 0$ we will define $f_n(x) = x^n \cdot g(x)$ from that $\{f_n\}$ is uniformly convergent in $[0.1]$
Proof:for all $x \in [0, 1)$ we have $f_n(x) \to_{n\to\infty} 0$ and $f_n(1) = 0$ therefore$f_n \overset{p}{\to} f \Rightarrow f = 0$

1. **First way** Let $\epsilon > 0$

> It is easy to see that for all $q < 1$ the sequence $\{x_n\}$ is uniformly convergent in the segment $[0, q]$ and we will notice that for $b_n \nearrow b, a_n \searrow a$ it is possible that $\{f_k\}$ is uniformly convergent for $[a_n, b_n]$ but not in (a, b)
> Will $\{x^n \cdot g(x)\}$ uniformly converge in $[0, q]$?

   (a) There exists a $\delta > 0$ such that for all $1 - \delta < x < 1$,$\left|g(x)\right| < \epsilon$ and in particular, for all n,

$$\left|f_n(x) - f(x)\right| = \left|f(x)\right| = |x|^n \cdot \left|g(x)\right| < |x|^n \cdot \epsilon < \epsilon$$

   (b) $g \in C[0, 1]$ therefore g is bounded by M:

$$\left|f_n(x)\right| < |x|^n \cdot M \overset{\forall x\in[0,1-\delta]}{\leq} (1 - \delta)^n \cdot M$$

   We will choose an N such that for all $n > N, (1 - \delta)^n \cdot M < \epsilon$ then for all $n > N$

$$\sup_{[0,1]}\left|f_n(x) - f(x)\right| = \max\{\sup_{[0,1-\delta]} f_n(x), \sup_{[1-\delta,1]} f_n(x)\} \overset{(2)}{<} \epsilon$$

   Note. *It is necessary to define $g(1) = 0$ or else we would have that $g \equiv 1$ which is a contradiction to the fact that*

2. **Second way**

> **Dini Theorem:** If we have $f_n : [a, b] \to \mathbb{R}$ which is a decending sequence($f_{n-1} > f_n$), $f_n \overset{p}{\to} f$ and both f_n and f are continuous. Then $f_n \overset{u}{\to} f$.
> Note that is is sufficient that for all points, $\{f_n(x)\}$ is monotonic.

(a) g is continuous $\Rightarrow f_n$ is continuous.

(b) $f \equiv 0$ is continuous.

(c) For all x:

$$f_{n+1}(x) = x^{n+1}, \qquad g(x) = x \cdot x^n, \qquad g(x) = x \cdot f_n(x)$$

$f_n(x) > 0$ or $0 > f_n(x)$ for all n, it depends on g.

We will assume WLOG $0 < g(x)$ $f_{n+1}(x) = x \cdot f_N(x) \overset{f_n(x)>0, x<1}{<} f_n(x) \Rightarrow \{f_n(x)\}$ is monotonously descending. From the Theorem, we can learn that the convergence is uniform.

7 Switching Limits

1.

$$\forall n. \lim_{x \to x_0} f_n(x) = a_n \qquad a_n \to a$$

$$\overset{?}{\Rightarrow} \lim_{x \to x_0} f(x) = a \qquad f_n \overset{p}{\to} f$$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \lim_{n \to \infty} f_n(x) \overset{?}{=} \lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = \lim_{n \to \infty} a_n = a$$

In general this is not true!, for example:

$$x_n = f_n(x) \qquad f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

$$\lim_{x \to 1} f(x) = 0 \qquad \lim_{n \to \infty} \lim_{x \to 1} f_n(x) = \lim n \to \infty 1 = 1$$

If there is uniform convergence then we can (Homework)

2.

$$\int_a^b f = \lim_{\lambda(\Pi) \to 0} \Sigma(f, \Pi, \tilde{t}) = \lim_{\lambda(\Pi) \to 0} \Sigma(\lim_{n \to \infty} f_n, \Pi, \tilde{t}) \overset{?}{=} \lim_{n \to \infty} \lim_{\lambda(\Pi) \to 0} \Sigma(f_n, \Pi, \tilde{t})$$

Allowed only if there is uniform convergence)

8 Question from a test 2014

$f_n : [0,1] \to [0,1]$ are convergent and $f_n \overset{u}{\to} f$
Prove that:

$$\frac{1}{n} \sum_{k=1}^{\infty} f_k \overset{u}{\to} f$$

In Calculus 1 we saw that

$$\{a_n\}, a_n \to a \Rightarrow \frac{1}{n} \sum_{k=1}^{\infty} a_k \to a$$

Proof $a_n \to a \Rightarrow$ there exists an N such that for all $n > N$: $|a_n - a| < \epsilon$ WLOG $a = 0$

$$\left| \frac{1}{n} \sum_{k=1}^{n} a_k \right| \leq \frac{1}{n} \sum_{k=1}^{N} a_k + \frac{1}{n} \sum_{k=N+1}^{n} |a_k| < \frac{M}{n} + \frac{1}{n}(n - N) \cdot \epsilon \overset{\text{for } n>>N}{<} \epsilon$$

Proof $f_n \overset{u}{\to} f$ there exists an N such that for all $n > N$

$$\sup_{[0,1]} |f_n - f| < \epsilon$$

WLOG assume $f \equiv 0$. $\{f_n\}$ are continuous therefore for all n ther eexists

$$M_n = \max_{[0,1]} |f_n| \qquad M := \max_{1 \leq n \leq N} M_n$$

$$(\star) \text{ For all } 1 \leq n \leq N \text{we have that } \max_{[0,1]} |f_n| \leq M$$

$$\sup_{[0,1]} \left| \frac{1}{n} \sum_{k=1}^{n} f_k(x) - f \right| \overset{f \equiv 0}{\leq} \frac{1}{n} \left[\sup_{[0,1]} \sum_{k=1}^{N} \left| f_k(x) \right| + \sup_{[0,1]} \sum_{k=N+1}^{n} \left| f_k(x) \right| \right] < \frac{1}{n} [M + (n - N) \cdot \epsilon] \overset{\text{For a large } n}{<} \epsilon$$