Calculus 2

Adi Gluckskin
Arazim

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1 The indefinite integral

1.1 So what did we learn?

$\int_a^b f$ where $-\infty < a < b < \infty$ and f is bounded.
Expansion

1. the segment is not infinite (e.g. $[0, \infty)$)
2. f is unbounded.

Notes:

1. If $f \in R[a,b]$ then $\int_a^b f(t)dt = \lim_{s \to b^-} \int_s^s f(t)dt$
2. If the problematic point $x_0 \in (a,b)$ then we will check $\int_a^b f = \int_a^{x_0} f + \int_{x_0}^b f$ and they only converge if f converges.
3. and if we are careful with ways of integration, then life is good!

1.2 Examples

1. $$\int_0^\infty \frac{1}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2} dx = \lim_{b\to\infty} \arctan|_0^b = \lim \arctan b = \frac{\pi}{2}$$
$$\int_{-\infty}^\infty \frac{1}{1+x^2} dx = \int_{-\infty}^0 ... + \int_0^\infty = \pi$$

2. $$\int_{-\infty}^\infty \frac{2xdx}{1+x^2} = \int_0^\infty \frac{dt}{1+t} = \ln t|_0^\infty$$
Which doesn't converge!!
$$\int_{-\infty}^\infty \frac{2xdx}{1+x^2} = \int_{-\infty}^0 + \int_0^\infty \frac{2xdx}{1+x^2} = \lim_{R_1\to\infty} \int_0^{R_1} \frac{2xdx}{1+x^2} + \lim_{R_2\to\infty} \int_{-R_2}^0 \frac{2xdx}{1+x^2}$$

3. $$\int_0^\infty e^{-x}dx = " - e^{-x}|_0^\infty = \lim_{b\to\infty} F(b) - F(0) = 1$$

4. $$\int_0^1 \frac{\log x}{x} dx = [t = \log x, x = e^t, dx = e^t dt, x = 0 \to t = -\infty, x = 1 \to t = 0] = \int_{-\infty}^0 \frac{t}{\cancel{e^t}} \cancel{e^t} dt = (\frac{t^2}{2}|_{-\infty}^0$$
which diverges!

1.3 Cauchy Theorem

$-\infty < a < b \leq \infty$ and $f \in R[a,t]$ for all $t < b$ then $\int_a^b$ converges iff

$$\forall \epsilon > 0. \exists B < b. \forall B < b_1 < b_2 < b \left| \int_{b_1}^{b_2} f \right| < \epsilon$$

f is continuous, non-negative so that $\int_0^\infty f$ converges. Find the limit $\lim_{n\to\infty} \int_0^1 f(nx)dx$

$$\int_0^1 f(nx)dx = [t = nx, dx = \frac{dt}{n}, x = 0 \to t = 0, x = 1 \to t = n] = \int_0^n f(t) \cdot \frac{dt}{n}$$

If we mark $a_n = \int_0^n f(t)dt$ then because $\int_0^\infty f$ converges then $\{a_n\}$ is a convergent sequence and in particular is bounded.

$$\lim_{n\to\infty} \int_0^1 f(nx) = \lim_{n\to\infty} \int_0^n f(t)dt = \lim_{n\to\infty} \frac{1}{n} \cdot a_n = 0$$

2 Convergence Tests

2.1 For keeping functions

1. $F = \int_0^x f, f \geq 0$ F does not rise and if F is bounded then $\int_0^\infty f$ converges.
2. Comparison test. Given $0 \leq g \leq f$ then:
   (a) if $\int g$ is divergent $\Rightarrow \int f$ is divergent.
   (b) if $\int f$ is convergent $\Rightarrow \int g$ is convergent.
3. the limit comparison test. Given f, g non-negative and lets assume that x_0 is a problematic point for both of them. if the limit $\lim_{x\to x_0} \frac{g(x)}{f(x)} = L$ exists then $L \in (0,\infty)$ then $\int f, \int g$ converge and diverge together.
   - $L = 0 \Rightarrow f > g$ from a certain point.
   - $L = \infty \Rightarrow g > f$ from a certain point.

   and we return to a normal comparison test.
4. Anchor function.
   - if $a > 0$ we will compare with $\int_a^\infty \frac{dx}{x^p}$. $p > 1$ converges. $p \leq 1$ diverges.
   - integrals of the type $\int_0^b \frac{dx}{x^p}(b > 0)$ converges if $p < 1$. diverges if $p \geq 1$.

2.2 Example problems

Check if the following integrals converge:

1. $\int_0^\infty \frac{x^2 dx}{x^4 - x^2 + 1}$ suspicious points: The only suspicious point is ∞. Divide and conquer - division of suspicious points of the function we are checking. it is no less important to make sure that for the function we are comparing in the range of which we are integrating there is only one suspicious point.

$$\int_0^\infty \frac{x^2}{x^4 - x^2 + 1} = \int_0^1 \frac{x^2 dx}{x^4 - x^2 + 1} + \int_1^\infty \frac{x^2 dx}{x^4 - x^2 + 1} = I_1 + I_2$$

I_1 is Riemann integrable and in particular is finite. for I_2 we will notice that $\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{x^2}{x^4-x^2+1}}$ and therefore $\frac{1}{x^2}$ and $\frac{x^2}{x^4-x^2+1}$ diverge and converge together! Which means that I_2 converges and the integral converges.

2. $\int_0^1 \frac{dx}{(\cos x-1)\sqrt{1-x}}$ We will notice that $\frac{1}{(\cos x-1)\sqrt{1-x}}$ keeps it sign and we may use comparison test. Suspicious points: the denominator vanishes at 0 and at 1 and therefore these are suspicious points of

$$\int_0^1 \frac{dx}{(\cos x-1)\sqrt{1-x}} = \int_0^{1/2} \frac{dx}{(\cos x-1)\sqrt{1-x}} + \int_{1/2}^1 \frac{dx}{(\cos x-1)\sqrt{1-x}} = I_1 + I_2$$

I_2: we will notice that $\lim_{x\to 1} \frac{\frac{1}{\sqrt{1-x}}}{\frac{1}{(\cos x-1)\sqrt{1-x}}}$ and the integrals converge and diverge together.

$$\int_{1/2}^1 \frac{-1}{\sqrt{1-x}} = \int_0^{1/2} \frac{1}{\sqrt{t}} dt < \infty$$

I_1: we will use a Taylor series, $\cos x = 1 - \frac{x^2}{2} + o(x^2)$

$$\lim_{x\to 0} \frac{\frac{-1}{x^2}}{\frac{1}{(\cos x-1)\sqrt{1-x}}} = \lim_{x\to 0} \frac{\frac{-1}{x^2}}{\frac{1}{(-\frac{x^2}{2}+o(x^2))\cdot\sqrt{1-x}}} = \lim_{x\to 0}(\frac{1}{2} + o(1)) \cdot \sqrt{1-x} = \frac{1}{2} \in (0,\infty)$$

from the comparison test, we get that I_1 converges iff $\int_o^{1/2} \frac{-1}{x^2}$ and the other diverges and therefore the integral diverges.

3. $\int_0^\infty \frac{|\sin x|}{x} dx$ We will mark: $I_k = \int_{\pi l}^{\pi(k+1)} \frac{|\sin x|}{x} dx$ $\sin x$ is uniformly continuous and therefore if there exists a

$$\delta > \left| x - (\pi k + \frac{\pi}{2}) \right| \Rightarrow \left| |\sin x| - \overbrace{\left| \sin(\pi k + \frac{\pi}{2}) \right|}^{=1} \right| < \frac{1}{2} \Rightarrow |\sin x| > \frac{1}{2}$$

$$I_k = \int_{\pi l}^{\pi(k+1)} \frac{|\sin x|}{x} dx = [y = x - \pi k] = \int_0^\pi \frac{|\sin y|}{y + \pi k} dy \ge \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \frac{|\sin y|}{y+\pi k} dy >$$

$$\ge \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \frac{1/2}{y+\pi k} dy > \frac{1}{2} \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \frac{1}{\pi(k+1)} = \frac{\delta}{\pi(k+1)}$$

$$\int_0^\infty \frac{|\sin x|}{x} dx = \sum_{k=0}^\infty I_k > \sum_{k=0}^\infty \frac{\delta}{\pi(k+1)} = \delta \cdot \sum_{k=1}^\infty \frac{1}{\pi k} = \infty$$

2.3 Absolute convergence

Problem: for which values $\alpha \ge 0$ the following integral absolutely converges?

$$\int_1^\infty \frac{\sin x}{x^\alpha} dx$$

We will look at $\int_1^\infty \frac{|\sin x|}{x^\alpha} dx$ because $|\sin x| \le 1$ we get $\frac{|\sin x|}{x^\alpha} \le \frac{1}{x^\alpha}$ and therefore if $\int_1^\infty \frac{1}{x^\alpha}$ converges, our integral converges too. For all $\alpha > 1$ $\int_1^\infty \frac{|\sin x|}{x^\alpha}$ absolutely converges. For $\alpha \le 1$ we will notice that

$$\forall x > 1 \frac{|\sin x|}{x^\alpha} \ge \frac{|\sin x|}{x}$$

and therefore from the comparison test and because $\int_1^\infty \frac{|\sin x|}{x} dx$ diverges, ours does too.

Another solution is

$$\int_1^\infty \frac{|\sin x|}{x^\alpha}dx = \frac{\sqrt{\sin^2 x}}{x^\alpha} \geq \frac{\sin^2 x}{x} = \frac{1}{2}(\frac{1-\cos 2x}{x}) = \overset{\text{Diverges}}{\frac{1}{2x}} - \overset{\text{Converges}}{\frac{\cos 2x}{2x}}$$

3 Abel and Dirichlet tests

Theorem: $f, g : [a, w) \to \mathbb{R}, w \in \mathbb{R} \cup \{\infty\}$ such that:

1. f is monotonous $f \in C^1[a, w)$
2. g is continuous.

Then:

- **Abel** - if f is bounded

$$\int_a^w g \text{ Converges } \Rightarrow \int_a^w f \cdot g \text{ Converges.}$$

- **Dirichlet** - $\lim_{x\to w} f = 0$ and $\int_a^x g = G(x)$ is a bounded function then $\int_a^w f \cdot g$ converges.

3.1 Problem from a test

1. Find the type of convergence of

$$\int_1^\infty \frac{\sin x}{x^\alpha}, \alpha \in \mathbb{R}$$

$\alpha > 1 \Rightarrow$ absolute convergence. $\alpha \leq 0$ then it diverges (problem for home). $\underline{0 < \alpha \leq 1}$: We have seen that there is no absolute convergence we will check the normal convergence using Abel-Dirichlet. Its requirements exist for

$$f(x) = \frac{1}{x^\alpha} \in C^1(1, \infty), \lim_{x\to\infty} f(x) = 0, g(x) = \sin x, \quad \overset{\text{Convergent} \Leftarrow \text{Bounded}}{G(x) = \pm \cos x}$$

2. Find the convergence of the following function:

$$\int_0^\infty \sin(x^2), \int_0^\infty \sin(x^2)dx = \int_0^1 \sin(x^2) + \int_1^\infty = I_1 + I_2$$

I_1 - Riemann, finite.

$$I_2 = \int_1^\infty \sin(x^2)dx = [t = x^2, \frac{dt}{2\sqrt{t} = dx}] = \frac{1}{2}\int_1^\infty \frac{\sin t}{\sqrt{t}}dt$$

which is not absolutely convergent from the previous function.

3.

$$\int_1^\infty \sin(x \cdot \log x)dx$$

$f(x) = x \log x, f'(x) = \log x + 1 > 1$ in the section $(1, \infty)$ f is monotonously rising and in particular is invertible. from a theorem, the inverse function f' is differentiable and in particular

$$(f^{-1}(y))' = \frac{1}{f'(f^{-1}(y))} \to_{y\to\infty} 0$$

$$\int_1^\infty \sin(f(X)) = [y = f(x)] = \int_0^\infty \sin(y) \cdot (f^{-1}(y))'dy = \int_0^\infty \frac{\sin y}{f'(f^{-1}(y))}dy$$

$2 \geq \left|\int_0^a \sin y\right|, \frac{1}{f'(f^{-1}(y))} \to 0$ from the diriclet test, the integral converges.

Another solution

$$\int_1^\infty \sin(f(x)) = \int_1^\infty \sin(f(x)) \cdot \frac{f'(x)}{f'(X)}$$

we will mark $g(x) =\sim (f(x)f'(x), h(x) = \frac{1}{f'(x)}$
$\int g(x) = -\cos(f(x))$ which is bounded! $\lim_{x\to\infty} \frac{1}{f'(x)} = 0$ which is mono' from the dirichlet test, we get that the integral is bounded!

4 The integral test

$f : [1, \infty) \to \mathbb{R}$ goes down, $f \geq 0$ and we will assume for all $f \in R[1, b]$ then the indefinite integral $\int_1^\infty f$ and the series $\sum_{n=1}^\infty f(n)$ converge and diverge together.

4.1 Examples

1. $\int_b^\infty \frac{dx}{x \ln \ln x}$, $f(x) = \frac{1}{x \ln \ln x}$ f keeps the requirements(monotonous) and we know from the cauchy condensation test that the series $\sum \frac{1}{x \ln \ln x}$ is a divergent series. therefore the integral is divergent.

2. Does the following sum converge or diverge?

$$\sum_{n=1}^\infty \frac{1}{n^\alpha}$$

$$\int_1^\infty \frac{1}{x^\alpha} \overset{\alpha \neq 1}{=} \frac{x^{1-\alpha}}{1-\alpha}|_1^\infty = ... = \begin{cases} \text{converges} & \alpha > 1 \\ \text{diverges} & \alpha < 1 \end{cases}$$

Why can we begin from $m > 1$ when using the integral test? f is monotonous $\Rightarrow$ in every segment $[1, m]$ whcih is Riemann integrable

$$\int_1^\infty = \overset{\text{finite}}{\int_1^m} f + \overset{\text{check}}{\int_m^\infty} f$$

which means the for every series, only the tail is interesting. Therefore you can start from $m > 1$.