Calculus 2

Adi Gluckskin
Arazim

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1 The definite integral

Definition: $f : [a, v] \to \mathbb{R}, \Pi = \{a = x_0 < x_1 < ... < x_n = b\}$ is a partition of the section $[a, b]$ $\Delta x_i = x_{i+1} - x_i$ for each i we will choose a $t_i \in [x_i, x_{i+1})$ and we will mark it with a $\vec{t} = (t_1, t_2, ..., t_n)$ a Riemann sum is

$$S(f, \Pi, \vec{t}) = \sum_{i=0}^{n} f(t_i) \cdot \Delta x_i \, , \delta(\Pi) = \max \Delta x_i$$

is the partition parameter(how refined it is). f will be called integrable over the interval [a,b] (and we will mark with $f \in R[a, b]$ if there exists an I such that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for each partition Π such that $\delta(\Pi) < \delta$ and for each choice of points $\vec{t}$ we get $\left|S(f, \Pi, \vec{t}) - I\right| < \epsilon$
Darboux criterion: $f : [a, b] \to \mathbb{R}$ is bounded. For each partition Π of $[a, b]$ we will define

$$m_i = \inf_{[x_i, x_{i+1}]} f \; M_i = \sup_{[x_i, x_{i+1}]} f \; \overline{\sum}(f, \Pi) = \sum_{i=0}^{n} M_i \cdot \Delta x_i$$

$$\underline{\sum}(f, \Pi) = \sum_{i=0}^{n} m_i \cdot \Delta x_i \; \overline{I} = \inf_{\Pi} \overline{\sum}(f, \Pi) = \inf\{\overline{\sum}(f, \Pi); \Pi \text{ is a division}\}$$

$$\underline{I} = \inf_{\Pi} \underline{\sum}(f, \Pi)$$

Definition - if f is bounded on $[a, b]$ then $f \in R[a, b] \Leftrightarrow \overline{I} = \underline{I}$
Oscillation: for the section $J \subset [a, b]$

$$\omega(f, J) = \sup_J f - \inf_J f$$

The oscillation of f in the section J if we mark $\omega_i = \omega(f, [x_i, x_{i+1}])$ then $\omega(f, \Pi) = \sum_{i=0}^{n} \omega_i \Delta x_i = \sum_{i=0}^{n} \Delta x_i (M_i - m_i) = \overline{\sum}(f, \Pi) - \underline{\sum}(f, \Pi)$ ω is the oscillation of f in relation to the partition Π in the section $[a, b]$.
Theorem f is Riemann integrable in $[a, b]$ iff f is bounded and $\forall \epsilon > 0. \exists \delta > 0. \forall \Pi. \delta(\Pi) < \delta \Rightarrow \omega(F, \Pi) < \epsilon$ or in other words, $\lim_{\delta(\Pi) \to 0} \omega(f, \Pi) = 0$

1.1 Examples

- The Dirichlet function: $D(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$ is it integrable?
  $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{R} \backslash \mathbb{Q}$ is dense in $\mathbb{R}$.therefore for each partition Π $M_i = 1, m_i = 0$, in other words $\omega_i = 1$. Overall, we get that f is not integrable because for all Π we have that $\omega(f, \Pi) = b - a = 1 \not\to 0$
  .

- Is the function $R:[0,1]\to\mathbb{R}$ $R(x)=\begin{cases}\frac{1}{q} & x=\frac{p}{q}\in\mathbb{Q}\\ 1 & \text{else}\end{cases}$ where $\frac{p}{q}$ is a reduced fraction, integrable?

  We will take an $\epsilon>0$ Π is a partition (we will choose the level of refinement at the end.)

$$B(f,\Pi)=\{I\in\Pi,\omega(f,I)\geq\frac{\epsilon}{2}\}$$

$$G(f,\Pi)=\{I\in\Pi,\omega(f,I)<\frac{\epsilon}{2}\}$$

$$\sum_{I\in B(f,\Pi)}\omega(f,I)\cdot|I|<1\cdot\sum_{I\in B(f,\Pi)}|I|\leq\delta\cdot\text{number of sections in }B(f,\Pi)\leq\delta N(\epsilon)\leq\frac{\epsilon}{2}$$

  Crucial point! it is important to check the order of choosing the parameters, in our case $N(\epsilon)$ is depended only on $\epsilon>0$
  Theorem: if $f,g\in R[a,b]$ if $f|_A=g|_A$ A is a dense set on $[a,b]$ then $\int_a^b f=\int_a^b g$ find $\int_0^1 R(x)$ - $R|_{\mathbb{R}\setminus\mathbb{Q}}=0$ and from the previous theorem $\int_0^1 R(x)=\int_0^1 0=0$

1.2 composition of integrable functions

if $f,g\in R[a,b]$ then $f\circ g$ is not necessarily integrable. $R(x)=g(x)\in R[0,1)$

$$f(x)=\begin{cases}1 & x=0\\ 0 & x\neq 0\end{cases}$$

$$f\circ g(x)=\begin{cases}0 & x\in\mathbb{Q}\\ 1 & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}=D(x)$$

which is not integrable!

1.3

if $g\in R[a,b]$ and $:\mathbb{R}\to\mathbb{R}$ is continuous then $f\circ g\in R[a,b]$
Intuition $g([a,b])\subset[-M,M]$ f is continuous $\Rightarrow$ f is uniformly continuous on $[-M,M]$

$$x,y\in I\in G(\Pi,g)\Rightarrow\big|g(x)-g(y)\big|<\delta\Rightarrow\big|f\circ g(x)-f\circ g(y)\big|<\epsilon$$

Given $\epsilon>0$ and Π such that $\lambda(\Pi)<\delta($ δ will be chosen in the future) because g is integrable for ϵ_1 there exists a δ such that for each partition Π $\lambda(\Pi)<\delta$ we have that $\omega(g,\Pi)<\epsilon_1$
Π is a partition as previously defined.

$$\omega(f\circ g,\Pi)=\sum_{I\in G(f\circ g,\Pi)}\omega(f\circ g,I)\cdot|I|+\sum_{I\in B(f\circ g,\Pi)}\omega(f\circ g,I)\cdot|I|=\sum_{I\in G(g,\Pi)}\omega(f\circ g,I)|I|+\sum_{I\in B(g,\Pi)}\omega(f\circ g,I)|I|$$

$$\sum_{I\in G(g,\Pi)}\omega(f\circ g)|I|<\frac{\epsilon}{2(b-a)}\cdot(b-a)=\frac{\epsilon}{2}$$

$g\in R[a,b]\Rightarrow$ there exists a $\delta>0$ such that $\lambda(\Pi)<\delta\Rightarrow\omega(g,\Pi)<\epsilon_1$

$$\sum_{I\in B(g,\Pi)}\omega(f\circ g,I)|I|<2L\sum_{I\in B(g,\Pi)}|I|\leq 2L\delta\cdot$$

$$\delta\cdot\sum_{I\in B(g,\Pi)}|I|<\sum_{I\in B(g,\Pi)}\epsilon_1\cdot|I|<\sum_{I\in B(g,\Pi)}\omega(g,I)\cdot|I|\leq\omega(g,\Pi)<\epsilon_2\Rightarrow\sum_{I\in B(g,\Pi)}|I|<\frac{\epsilon_2}{\epsilon_1}$$

.
The order of choosing the parameters: given $\epsilon > 0$ because of teh uniform continuity, we can choose an ϵ_1 such that $|x-y| < \epsilon_1 \Rightarrow \big|f(x) - f(y)\big| < \frac{\epsilon}{x(b-a)}$
we will choose an ϵ_2 such that $\frac{\epsilon_2}{\epsilon_1} < \frac{\epsilon}{4L}$ and we will choose the fineness δ by $\delta = min\{\Delta_i, \epsilon_1\}$ Δ_1 such that $\lambda(\Pi) < \Delta_1 \Rightarrow \omega(g, \Pi) < \epsilon_2$

1.4 The composition of integrable functions

From the previous section, we learn that $f \in R[a,b] \Rightarrow e^f, \sqrt{f}, \sin f...$ are integrable even though we do not always have a elementary anti-derivative.

1.5 Intermediate value property of the integral

Theorem $f : [a,b] \to \mathbb{R}$ is integrable and continuous then there exists a $c \in (a,b)$ such that $\frac{1}{a-b}\int_a^b f = f(c)$

1.5.1 Question from a test!

- $f, g : [-1,1] \to \mathbb{R}$ are continuous g is even and $\int_{-1}^{1} f = \int_0^1 g$ prove that there exists an $x_0 \in (-1,1)$ such that $2 \cdot f(x_0) = g(x_0)$
- IS it necessary that g, f are continuous?

Answer:

- We will define $h(x) = g(x) - 2f(x)$ h is continuous and in particular it is integrable. From the intermediate integrability property, there exists an $x_0 \in (-1,1)$ such that

$$h(x_0) = \frac{1}{1-(-1)}\int_{-1}^{1} h(x) = \frac{1}{2}\int_{-1}^{1} g(x) - 2f(x)dx =$$

$$\frac{1}{2}(\int_{-1}^{1} g(x) - 2\int_{-1}^{1} f(x)) = \frac{1}{2}(2\int_0^1 g(x) - 2\int_{-1}^{1} f(x)) = \int_0^1 g(x) - \int_{-1}^{1} f(x) = 0$$

  there exists an x_0 such that $h(x_0) = 0 \Rightarrow g(x_0) = 2f(x_0)$
- If they aren't continuous it doesn't necessarily happen!In this case: $f = \frac{3}{8}$ f is continuous$g = \begin{cases} \frac{1}{2} & |x| < \frac{1}{2} \\ 1 & \text{else} \end{cases}$

$$\int_0^1 g = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2} = \frac{3}{4}\int_{-1}^{1} f = \frac{3}{4} \Rightarrow \frac{3}{8}\cdot 2 = \frac{6}{8} \notin \{\frac{1}{2}, 1\}$$

2 The fundamental theorem of calculus

If $f \in R[a,b]$ and $x_0 \in [a,b]$ where f is continuous then the function $F(x) = \int_a^x f(t)dt$ F is differentiable in x_0 and we have that $F'(x_0) = f(x)$ if in addition to that f has an anti-derivative in $[a,b]$ then

$$\int_a^b f(t)dt = F(b) - F(a)$$

in particular if it has an anti-derivative F then it is differentiable in $[a,b]$ and the definitions combine.

2.1 Note

We need two things, that $f \in R[a, b]$ and that f has an anti-derivative. They are both needed. In essence, if F is differentiable but $f = F'$is unbounded then f is not integrable,

$$f(x) = \begin{cases} x^2 \cdot \sin(\frac{1}{x^2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

2.2 Problems

- find the derivative of the function

  $$F(x) = \int_0^{2x} f \cdot e^t dt$$

  We will mark with $f(x) = \int_0^x t \cdot e^t dt$ and from the fundamental theorem of calculus, we get the f is differentiable, and that $f'(x) = x \cdot e^x$ currently,we will notice that $F(x) = f(2x), F'(x) = f'(x) = f'(2x) \cdot 2 = 2xe^2x \cdot 2$

2.3 Ways of integration:

Integration by Parts: if we have $u, v \in C^1[a, b]$ $\int_a^b u' \cdot v = u \cdot v|_a^b - \int_a^b v' \cdot u$ Note We have stated continuity and not just differentiability, because we wanted that u', v' will be integrable. in the same way we could have checked that u, v are differentiable and that u', v' are integrable.

1. Find - $\int_1^e \ln x dx = F(e) - f(1) = 1$
2. $\int_0^{2\pi} x \cdot \sin x dx = -x \cos x|_0^{2\pi} - \int_0^{2\pi} - \cos x = ... = -2\pi$

Changing variables: if f is continuous and $\varphi \in C^1$ then $\int_a^b f(x) = \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t) dt$

1. f is continuous such that $f \circ \varphi \in R \in [\alpha, \beta]$
2. $\forall t \in [\alpha, \beta], \varphi(t) \in [a, b]$
3. φ is injective by segment $\int = \int + \int + \int ...$

$$\int_0^{\Pi/4} \frac{\cos x}{1 + (\sin x)^2} = \int_0^{\Pi/4} \frac{d(\sin x}{1 + \sin^2 x} = \int_?^? \frac{1}{1 + t^2} dt = \arctan t = \arctan(\sin(x))|_0^{\pi/4}$$

2.3.1 Notes

When changing the variables we can change the limits of the integration and we do not have to reverse the variables.
Sometimes it wont be worth it to find the new limits of integration and it may be easier to reverse the variables.

2.4 Another use for integrals

Finding Limits:

$$\lim_{n \to \infty} \sum_{k=1}^n \frac{k}{n^2} = \sum_{k=1}^n \frac{1}{n} \cdot f(\frac{k}{n})$$

In thsi case we have $\Pi = \{0 < \frac{1}{n} < \frac{2}{n} < ... < \frac{n}{n}\}$ in our case $f(x) = x) if t_k^n = \frac{k}{n} = x_k$ Therefore we have

$$\lim_{n \to \infty} \sum_{k=1}^n \frac{k}{n^2} = \lim_{\lambda(\Pi_n) \to 0} S(x, \Pi_n, \vec{t^n} = \int_0^1 x dx = \frac{1}{2}$$

Another problem: $f:[-a,a]\to\mathbb{R}$ is integrable and odd, therefore $\int_{-a}^{a} f = 0$
First way: if Π is a divsion of the section $[0,a]$ and we create the partition $\tilde{\Pi}=\{x_i,-x_i|x_i\in\Pi\}$ (0 does not bother us becasue $f(0)=0$

$$S(f,\tilde{\Pi},\vec{t})$$

given a choice of points $t_i\in[x_i,x_{i+1})$ of Π we will define $\tilde{t}$. Given $\vec{t}$ a choice of points from the partition Π

$$\tilde{t_i}=\begin{cases} t_i & I_i\in\Pi \\ -t_i & \text{else} t_i\in I_{-i}\end{cases}$$

$$S(f,\tilde{\Pi},\tilde{\vec{t}})\to_{\lambda(\tilde{\Pi})\to 0} I\sum_{i=1}^{n}\Delta x_i\cdot f(\tilde{t_i})=\sum_{i=1}^{n}\Delta x_i(f(t_i)+f(-t_i))$$

Another way: the idea - Changing variables! Problem, f is integrable but not necessarily continuous. Claim(from home): if $\varphi = ax+b$ then changing variables is legal.