Calculus 2

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1 Before you start

Definition: a function $f = \frac{P}{Q}$ where P and Q are polynomials is called a rational function. When $\deg Q > \deg P$ then f is called a simple rational function. If on top of that, f is of the following types:

1. $\frac{A}{(x-a)^n} n \geq 1, A, a \in \mathbb{R}$
2. $\frac{Ax+B}{(x^2+px+q)^n}, p^2 - 4q < 0, A, B \in \mathbb{R}, n \geq 1$

Then f is an elementary rational function.

1.1 Working stages

1. Changing a general function to a rational function.
2. Moving from a certain rational function to a simple rational function.
3. Moving from a simple rational function to an elementray rational function.
4. Solving the integral of elementary rational functions.

1.2 Claim

The following always happen

1. $$\int \frac{A}{x-a} = A \cdot \ln|x-a| + c$$
2. $$n > 1, \int \frac{A}{(x-a)^n} = \frac{A}{(1-n)(x-a)^{n-1}} + c$$
3. $$\int \frac{Ax+B}{x^2+px+q} \overset{p^2-4q<0}{=} \frac{A}{2} \ln(x^2+px+q) + \frac{B - \frac{A \cdot q}{2}}{\sqrt{q - (\frac{p}{2})^2}} \cdot \arctan(\frac{x + \frac{p}{2})}{\sqrt{q - (\frac{p}{2})^2}} + c$$
4. $$I_n = \int \frac{1}{(x^2+px+q)^n} \Rightarrow I_{n+1} = \frac{x + \frac{p}{2}}{2n(q - (\frac{p}{2})^2)(x^2+px+q)^n} + \frac{2n-1}{2n(q - (\frac{p}{2})^2)} \cdot I_n$$
5. $$\int \frac{Ax+B}{(x^2+px+q)^n} = \frac{A}{2(1-n)(x^2+px+q)^{n-1}} + (B - \frac{A \cdot p}{2}) \cdot I_n$$

1.3 Proof

1.4 Moving from a simple rational function to a sum of elementary ones

<u>Order of operations:</u> $R(x) = \frac{P(x)}{Q(x)}$

1. We will split $Q(x)$ to a product of irreducible items.(The items are: $(x-a)^n, (x^2+px+q)^n, p^2-4q < 0$
$$Q(x) = Q_1 \cdot Q_2 \cdot ... \cdot Q_n$$
2. for every item of the type $Q_i = (x-a)^k$ we will write the sum $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + ... + \frac{A_n}{(x-a)^k}$ and for every item of the type $Q = (x^2+px+q)^k$ we will write the sum: $\frac{A_1+B_1}{x^2+px+q} + \frac{A_2+B_2}{(x^2+px+q)^2} + ... + \frac{A_k+B_k}{(x^2+px+q)^k}$

   If $degP < degQ$ then we will split the denominator. $\frac{P}{Q} = \frac{P}{Q_1 \cdot Q_2} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \overset{\frac{1}{Q_1}}{...} + \frac{A_n}{(x-a)^k} + \frac{A_1+B_1}{x^2+px+q} + \overset{\frac{1}{Q_2}}{\frac{A_2+B_2}{(x^2+px+q)^2}} + ... + \frac{A_l+B_l}{(x^2+px+q)^l}$

1.5 Examples

1. $\frac{1}{x^2-1}, P = 1, Q = x^2+1 = (\overset{Q_1}{x+1}) \cdot (\overset{Q_2}{x-1}), \frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1) = x(A+B) + B - A \Rightarrow \begin{cases} A+B=0 \\ B-A=1 \end{cases} \Rightarrow A = -B = -\frac{1}{2} \Rightarrow \frac{1}{x^2-1} = \frac{-1}{2(x-1)} + \frac{1}{2(x+1)}$
2. $\frac{x+3}{\underset{Q_1}{(x-1)^2} \cdot \underset{Q_2}{(x+1)^3}} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2} + \frac{B_3}{(x+1)^3} \Rightarrow x+3 = (x-1)(x+1)^3 A_1 + A_2(x+1)^3 + B_1(x+1)^2(x-1)^2 + (x+1)(x-1)^2 B_2 + (x-1)^2 B_3$
   <u>$x = 1$:</u> $4 = A_2 \cdot 8 \Rightarrow A_2 = \frac{1}{2}$
   <u>$x = (-1)$:</u> $2 = B_3 \cdot 4 \Rightarrow B_3 \frac{1}{2}$
   $x \in \{\frac{1}{2}, -\frac{1}{2}, 0\}$
   Answer:
$$\frac{x+3}{(x+1)^3(x-1)^2} = -\frac{5}{8} \cdot \frac{1}{x-1} + \frac{1}{2} \cdot \frac{1}{(x-1)^2} + \frac{5}{8}\frac{1}{x+1} + \frac{3}{4} \cdot \frac{1}{(x+1)^2} + \frac{1}{2} \cdot \frac{1}{(x+1)^3}$$
3.
$$\frac{1}{(x-1)(x^2+1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (x-1)(Bx+c) =$$
$$x^2(A+B) + x(C-B) + A - C \Rightarrow \begin{cases} A+B=0 \\ C-B=0 \\ A-C=1 \end{cases} \Rightarrow A = \frac{1}{2}, B = C = -\frac{1}{2}$$

1.6 Moving from a rational function to a simple rational function

In order to move from a general rational function to a simple function, we will use long division.

1.
$$\frac{x^3-2x-1}{x^2+3} = x - \frac{5x+1}{x^2+3}$$
$$\begin{array}{r|l} & x \\ \hline x^2+3) & x^3-2x-1 \\ & -x^3-3x \\ \hline & -5x \end{array}$$

2.

$$\frac{x^3+1}{x+1} = x^2 - x + 1$$

$$\begin{array}{r|l} & x^2 - x + 1 \\ \hline x+1 & x^3 \qquad\quad + 1 \\ & -x^3 - x^2 \\ \hline & \quad -x^2 \\ & \quad x^2 + x \\ \hline & \qquad x + 1 \\ & \qquad -x - 1 \\ \hline & \qquad\quad 0 \end{array}$$

1.7 Full example

$$\int \frac{x^4}{x^3+x^2-x-1} = ?$$

<u>Stage 2</u>

$$\begin{array}{r|l} & \qquad\qquad x - 1 \\ \hline x^3+x^2-x-1 & x^4 \\ & -x^4 - x^3 + x^2 + x \\ \hline & \quad -x^3 + x^2 + x \\ & \quad x^3 + x^2 - x - 1 \\ \hline & \quad 2x^2 \qquad -1 \end{array}$$

$$\int \frac{x^4}{x^3+x^2-x-1} = x - 1 + \frac{2x^2-1}{x^3+x^2-x-1}$$

<u>Stage 3</u> Guiding line we will find the roots and divide $\overline{Q|}(x-a)$ for the root a.

$$\begin{array}{r|l} & \qquad x^2 + 2x + 1 \\ \hline x-1 & x^3 \; + x^2 \; - x - 1 \\ & -x^3 \; + x^2 \\ \hline & \quad 2x^2 \; - x \\ & \; -2x^2 + 2x \\ \hline & \qquad x - 1 \\ & \qquad -x + 1 \\ \hline & \qquad\quad 0 \end{array}$$

$= (x+1)^2(x-1)$

$$\frac{2x^2-1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2} \Rightarrow 2x^2 - 1 =$$
$$A(x+1)^2 + B_1(x^2-1) + B_2(x-1) = x^2(A+B_1) + x(2A+B_2) + A - B_1 - B_2$$

We will compare the coefficients

$$\begin{cases} a + B_1 = 2 \\ 2A + B_2 = 0 \\ A - B_1 - B_2 = (-1) \end{cases} \Rightarrow A = \frac{1}{4}, B_1 = \frac{7}{4}, B_2 = -\frac{1}{2}$$

Stage 4:

$$\int \frac{x^4}{x^3+x^2-x-1} = \int x-1+\frac{1}{4}\int \frac{1}{x+1} - \frac{1}{2}\int \frac{1}{(x+1)^2} = ...$$

1.8 Example from a test

$$f(x) = \frac{x^3}{(x^2-1)(x+1)}$$

Stage 2:

$$Q = (x^2)(x+1) = (x+1)^2(x-1) = x^3+x^2-x-1$$

$$\begin{array}{r|l} & 1 \\ x^3+x^2-x-1 & x^3 \\ & -x^3-x^2+x+1 \\ \hline & -x^2+x+1 \end{array}$$

$$f(x) = 1 - \frac{x^2-x-1}{x^3+x^2-x-1}$$

Stage 3:

$$\frac{x^2-x-1}{x^3+x^2-x-1} = \frac{x^2-x-1}{(x^2-1)(x+1)} = \frac{A}{x-1} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2} \Rightarrow x^2(A+B_1)+x(2A+B_2)+A-B_1-B_2$$

$$\begin{cases} A+B_1 = 1 \\ 2A+B_2 = -1 \\ A-B_1-B_2 = -1 \end{cases}$$

$$f(x) = \frac{A}{x-1} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2}$$

$$\lim_{x\to 1} f(x)\cdot(x-1) = \frac{1}{4} = A$$

$$\lim_{a\Rightarrow -1} f(x)\cdot(x+1)^2 = \frac{1}{2} = C$$

$$\lim_{x\to 1}(f(x) - \frac{1}{2}\cdot\frac{1}{(x+1)^2})^{x+1} = ... = \frac{5}{4} = B$$

$$\int f(x) = \int 1 + \frac{1}{4}\int \frac{1}{x-1} \pm \frac{5}{4}\int \frac{1}{x+1} + \frac{1}{2}\int \frac{1}{(x+1)^2}$$

1.9 Moving from a general function to a rational function

Signs: we will mark with $R(f_1(x), f_2(x)...f_k(x))$ a rational function where f_j are its variables.
for example:

$$h = \frac{\cos x + 1}{\sin^2 x + \cos x + 7}, R = \frac{x+1}{y^2+x+7}, f_1 = \cos x, f_2 = \sin x, h = R(f_1, f_2)$$

1.9.1 example

1.

$$f = R(x,^m \sqrt{\frac{ax+b}{cx+d}}), t =^m \sqrt{\frac{ax+b}{cx+d}}, x(t) = \frac{b - t^m \cdot d}{t^m \cdot c - a} \int R(x,^m \sqrt{\frac{ax+b}{cx+d}}) = R(\frac{b - t^m \cdot d}{t^m \cdot c - a}, t) \cdot x'(t)dt$$

because $x(t)$ is a rational function so also $x'(t)$ therefore a rational function.
For example:

$$\int \frac{x \cdot^3 \sqrt{\frac{x+1}{x-1}}}{2x +^3 \sqrt{\frac{x+1}{x-1}}} = [t =^3 \sqrt{\sqrt{\frac{x+1}{x-1}}}, x = \frac{1+t^3}{t^3-1}, dx = \frac{3t^2}{(t^3-1)^2}] = \int \frac{\frac{1+t^3}{t^3-1} \cdot t}{2 \cdot \frac{1+t^3}{t^3-1} + t} \cdot \frac{3t^2}{(t^3-1)^2} dt$$

another example: $\frac{1}{1+\sqrt{x}} = \int \frac{1}{1+t} \cdot 2tdt = ...$

2.

$$f = R(x,^{m_1} \sqrt{\frac{ax+b}{cx+d}}, ...,^{m_k} \sqrt{\frac{ax+b}{cx+d}}) m = \prod_{j=1}^{k} m_j, t =^m \sqrt{\frac{ax+b}{cx+d}},^{m_j} \sqrt{\frac{ax+b}{cx+d}} = t^{\frac{m}{m_j}}, m = lcm(m_1, ..., m_k)$$

3. $f = R(x, \sqrt{ax^2 + bx + c})$

- if a¿ 0

$$\sqrt{ax^2+bx+c} = t - \sqrt{a} \cdot x \Leftrightarrow \cancel{ax^2} + bx + c = t^2 - 2tx\sqrt{a} + \cancel{ax^2} \Rightarrow x = \frac{t^2 - c}{b + 2\sqrt{x} \cdot t}$$

- if c ¿ 0

$$\sqrt{ax^2+bx+c} = x \cdot t + \sqrt{c}$$

- if there are two roots :

$$x^2+bx+c = a(x-x_1)(x-x_2) \Leftrightarrow \sqrt{ax^2+bx+c} = \sqrt{a} \cdot \sqrt{(x-x_1)(x-x_2)} = \sqrt{a}(x-x_1) \cdot \sqrt{\frac{x-x_2}{x-x_1}}$$

4.

$$f(x) = R(\sin x, \cos x) t = \tan(\frac{x}{2}), x = 2 \arctan t \begin{cases} \cos x = \frac{1-tan^2(\frac{x}{2})}{1+tan^2(\frac{x}{2}} = \frac{1-t^2}{1+t^2} \\ \sin x = \frac{2\tan(\frac{x}{2})}{1+\tan(\frac{x}{2})} = \frac{2t}{1+t^2} \end{cases}$$

5.

$$\int R(x, \sqrt{a^2 - x^2}), x = \arcsin t$$

6.

$$\int R(x, \sqrt{x^2 - a^2}), x = \frac{a}{\cos t}$$

7.

$$\int R(x, \sqrt{x^2 + a^2}), x = a \cdot \tan t$$

Simple example:

$$\int \frac{dx}{sin^3 x} = \int \frac{1}{(\frac{2t}{a+t^2})^3} \cdot \frac{2}{1+t^2} dt, t = \tan x$$

Final example:

$$\int \frac{x^2}{\sqrt{9+x^2}} = \int \frac{9\tan^2 t}{\sqrt{9 + 9\tan^2 t}} \cdot \frac{3}{\cos^2 t} dt = \int \frac{9\frac{\sin^2 t}{\cos^2 t}}{\sqrt{1 + \frac{\sin^2}{\cos^2 t}}} \cdot \frac{1}{\cos^2 t} dt = 9 \int \frac{\sin^2 t}{\cos^3 t} dt = 9 \int R(\sin x, \cos x)$$