Calculus 2

Adi Gluckskin
Arazim

March 8, 2015

1 Before you start

- Email: adigluck@post.tau.ac.il.
- Reception hours: after the lesson on monday.
- Homework is not mandatory.

2 Reminders from Calc 1

2.1 Darboux Theorem [Intermediate value propery of the derivative]

if f is differentiable in (a,b) = I then f' achieves every value between $f'(a)$ and $f'(b)$.
- Not all differentiable functions are continuosly differentiable.

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

2.2 Definition

$f : I \to R$ is called antiderivative of f if for all $x \in I f(x) = F'(x)$. if F_1, f_2 are antiderivatives of f then $F_1 = F_2 + c$ where c is a constant.

3 Basic Integrals

1. $$\int x = \frac{x^2}{2} + c$$
2. $$\int x^\alpha = \frac{x^{\alpha+1}}{\alpha + 1} + c$$
3. $$\int \frac{1}{x} = \ln x + c$$
4. $$\int e^x = e^x + c$$

5.

$$\int \sin x = -\cos x + c$$

6.

$$\int \frac{1}{\cos^2 x} = \tan x + c$$

7.

$$\int \ln x = x \ln x - x + c$$

8.

$$\int \frac{1}{1+x^2} = \arctan x + c$$

4 Problem

find the antiderivative (if it exists) of the following functions , if they do not exist explain why.

1. Derivative: $f : (0, 1) \cup (1, 2) \to R$

$$f(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 0 & \text{if } x \in (2,3) \end{cases}$$

Antiderivative: $F : (0, 1) \cup (1, 2) \to R$

$$F(x) = \begin{cases} \frac{x^2}{2} + c_1 & \text{if } x \in (0,1) \\ c_2 & \text{if } x \in (2,3) \end{cases}$$

2. Derivative: $f : (0, 1) \cup [1, 2) \to R$

$$f(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 0 & \text{if } x \in [1,2) \end{cases}$$

Answer: this is not a integrable function because it does not have the Darboux property. In the section $(\frac{1}{2}, \frac{3}{2}) \in (0, 2)$ we have the two extreme points $f\frac{1}{2} = \frac{1}{2}f(\frac{3}{2}) = 0$ However, the value 1/4 is not in the segment $(\frac{1}{2}, \frac{3}{2})$.

5 Theorem

If f is continuous in the section I then it is integrable in that section.

5.1 claim

if f,g are integrable then $f + -g$, $\alpha \cdot f \alpha \in R$

5.2 problem

Find the anti-derivative function of the following functions:

1.

$$\int (\frac{1-x}{x})^2 = \int \frac{1-2x+x^2}{x^2} = \int \frac{1}{x^2} - 2\int \frac{1}{x} + \int 1 =$$
$$\frac{-1}{x} + c_1 - 2(\ln|x| + c_2) + x + c_3 = \frac{-1}{x} - 2\ln|x| + x + c$$

2.

$$\int \frac{1}{\sqrt{x+1}+\sqrt{x-1}} = \frac{\sqrt{x+1}+\sqrt{x-1}}{(x+1)-(x-1)} = \int \frac{\sqrt{x+1}+\sqrt{x-1}}{2} =$$
$$\frac{1}{2}(\int (x+1)^{\frac{1}{2}} \int (x-1)^{\frac{1}{2}}) = \frac{1}{2}\cdot\frac{2}{3}\cdot((x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}}) + c$$

3.

$$\int \frac{x^3-1}{x-1} = \int \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = \int x^2 + \int x + \int 1 = \frac{x^3}{3} + \frac{x^2}{2} + x + c$$

4.

$$\int \sin^2 x = [\frac{\sin^k x}{\cos^k x}\frac{\phi(\sin(kx))}{\phi(\cos(kx))}] \Rightarrow \sin^2 x = \frac{1-cos(2x)}{2} =$$
$$\int \frac{1-cos(2x)}{2} = \int \frac{1}{2} - \frac{1}{2}\int \cos(2x) = \frac{x}{2} - \frac{sin(2x)}{4} + c$$

6 Theorem

f is a differentiable function on I then $\int \frac{f'}{f} = \ln|f| + c$

6.1 Example

$$\int \frac{x}{1+x^2} = \frac{1}{2}\int \frac{2x}{1+x^2} = \frac{1}{2}\int \frac{u'}{u} = \frac{1}{2}\ln|u| + c = \frac{\ln(1+x^2)}{2} + c$$

6.2 Symmetric Theorem

if f is differentiable on I then $\int f' \cdot e^f = e^f + c$

6.2.1 example

$$\int x \cdot e^{\frac{1}{2}x^2} = \int v' e^v = e^{\frac{x^2}{2}} + c$$

7 The non exact integral

There are 2 types of integration

7.1 Integration in parts

Claim - u,v: $I \to R$ are differentiable then $\int u' \cdot v = u \cdot v - \int u \cdot v'$

7.1.1 example

Find the antiderivative function of the following functions:

1. $$\int x \cdot \cos x = x \cdot \sin x - \int \sin x = x \cdot \sin x + \cos x + c$$

2. $$\int xe^x = x \cdot e^x - e^x + c$$

3. $$\int \ln x = \int 1 \cdot \ln x = x \ln x - \int 1 = x \ln x - x + c$$

4. $$\int x^3 \cdot \ln x = \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} = \frac{x^4}{4}(\ln x - \frac{1}{4}) + c$$

5. $$\int x^3 \cdot \ln^2 x = \frac{x^4 \ln x}{4} - \frac{1}{2} \int x^3 \cdot \ln x \text{ ... See previous problem.}$$

6. $$\int \arctan x = \int 1 \cdot \arctan x = x \cdot \arctan x - \int \frac{x}{1+x^2} = x \cdot \arctan x - \frac{1}{2} \ln(1+x^2) + c$$

7. $$I = \int e^x \cdot \cos x = e^x \cdot \cos x + \int e^x \sin x = e^x \cos x + (e^x - \int e^x \sin x - \int e^x \cos x)$$
$$\Rightarrow 2I = e^x(\cos x + \sin x) + c \Rightarrow I = \frac{e^x}{2}(\cos x + \sin x) + c$$

In a similar fashion to num 5 we can find the preivous function by recursion of the types:

1. $$P_n(x) \cdot \ln^k(x)$$

2. $$P_n(x) \cdot e^x$$

3. $$P_n(x) \cdot \cos x$$

7.2 Replacement of variables

Special case: $f : I \to R$, given F which is the antiderivative of f, can we find what the antiderivative of $f(a \cdot x + b)$ is?

$$\int f(ax+b) = \frac{F(ax+b)}{a} + c$$

7.2.1 Theorem

If the function f has an antiderivative F and g is differentiable, then:

$$F(g(x)) = \int f(g(x)) \cdot g'(x)dx$$

an agreed upon way of writing is:

$$\int f(t)dt = \int f(g(x)) \cdot g'(x)dx$$

7.2.2 Examples

1.

$$\int (x-1)^{100} = (f(t) = t^{100}, g(x) = x-1) f(g(x)) \cdot g'(x)dx =$$
$$f(t)dt = \int t^{100}dt = \frac{t^{101}}{101} + c = \frac{(x-1)^{101}}{101} + c$$

2.

$$\int (2x+1)^{10} = \frac{1}{2}\int f(g(x)) \cdot g'(x)dx = \frac{1}{2}\int f(t)dt = \frac{t^{11}}{11} + c = \frac{(2x+1)^{11}}{22} + c$$

3.

$$\int x^2(x^3+2)^{\frac{1}{5}} = \frac{1}{3}\int f(g'(x)) \cdot g'(x)dx = \frac{1}{3}\int f(t)dt = \frac{1}{3} \cdot \frac{5}{6} \cdot t^{frac65} + c = \frac{5}{18}(x^3+2)^{\frac{6}{5}} + c$$

4.

$$\int \frac{dx}{x \ln x} = \int f(g(x))g'(x) = \int f(t) = \ln|t| + c = \ln|\ln(x)| + c$$

5. In the open segment $(0, \frac{\pi}{2})$:

$$\int \frac{dx}{\sin x \cos x} = \int \frac{1}{\cos^2 x} \cdot \frac{1}{\frac{\sin x}{\cos x}}dx = \int \frac{1}{\cos^2 x} \cdot \frac{1}{\tan x}dx = \int f(g(x)) \cdot g'(x)dx = \quad (1)$$
$$\int f(t)dt = \int \frac{1}{t}dt \ \ln|t| + c = \ln|\tan x| + c = \ln \tan x + c \quad (2)$$

Note: a way to remember the equation from the first example is the following trick $\frac{dt}{df} = f'$ for $t = g(x) \to dt = g'(x)dx$ In the other direction, we must check that g is invertible (in the end we will get the antiderivative in t and we will want to return to x).

7.2.3 Examples

1.

$$\int \frac{dx}{(1+x)\cdot\sqrt{x}} = \int \frac{1}{(t^2+1)t} \cdot 2t \cdot dt = 2\int \frac{1}{t^2+1}dt = 2\arctan t + c = 2\arctan\sqrt{x} + c$$

2.

$$\int \frac{dx}{x \cdot \sqrt{x^2-1}} = \int \frac{1}{\sqrt{t^2+1}\cdot t} \cdot \frac{t}{\sqrt{t^2+1}}dt = \int \frac{dt}{t^2+1} = \arctan t + c = \arctan\sqrt{x^2-1}$$

x is defined on $(-\infty, -1) \cup (1, \infty)$ therefore x is defined on R.

3.

$$\int \frac{dx}{x\cdot\sqrt{x^2-1}} = \int \frac{t}{\sqrt{\frac{1}{t^2}-1}}\cdot\frac{-1}{t^2}dt = \int \frac{t\cdot\sqrt{t^2}}{\sqrt{1-t^2}}\cdot\frac{-1}{t^2}dt = -\int\frac{dt}{\sqrt{1-t^2}} =$$

$$\int\frac{dt}{\sqrt{1-t^2}} = \begin{cases}\arccos t + c_1 & \text{if } x>0\\ \arcsin t + c_2 & \text{if } x<0\end{cases} = \begin{cases}\arccos \frac{1}{x} + c_1 & \text{if } x>0\\ \arcsin \frac{1}{x} + c_2 & \text{if } x<0\end{cases}$$