Basic Combinatorics - Spring '13 — Instructor: Asaf Shapira

Home Assignment 5 — **Due Date: 28/05/13**

Please submit organized and well written solutions!

Problem 1. Prove that the number of surjective (i.e. onto) mappings from $[n]$ to $[k]$ is given by $\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$. Use this to deduce that:

- $\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^n = n!$.
- $\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = 0$ when $k > n$.
- $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$, where $S(n,k)$ are the Stirling numbers of the second kind.

Problem 2. Consider the number of ways of coloring the integers $\{1,..,2n\}$ using the colors red/blue in such a way that if i is colored red then so is $i-1$. Deduce the identity

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}2^{2n-2k} = 2n+1$$

Problem 3. Let $\varphi(n)$ denote the Euler totient function.

1. Show that if m,n are coprime then $\varphi(mn) = \varphi(m)\varphi(n)$ (you can use the formula for $\varphi(n)$).
2. Derive the formula of $\varphi(n)$ from the assumption that $\varphi(mn) = \varphi(m)\varphi(n)$ for coprime m,n.
3. Use item (1) to prove that $\sum_{d|n}\varphi(d) = n$.

Problem 4. Let $A_1,\dots,A_n$ be a family of n sets. Show that

$$|\bigcup_{i=1}^{n} A_i| \geq \sum_{1\leq i\leq n}|A_i| - \sum_{1\leq i<j\leq n}|A_i\cap A_j|$$

and

$$|\bigcup_{i=1}^{n} A_i| \leq \sum_{1\leq i\leq n}|A_i| - \sum_{1\leq i<j\leq n}|A_i\cap A_j| + \sum_{1\leq i<j<k\leq n}|A_i\cap A_j\cap A_k|$$