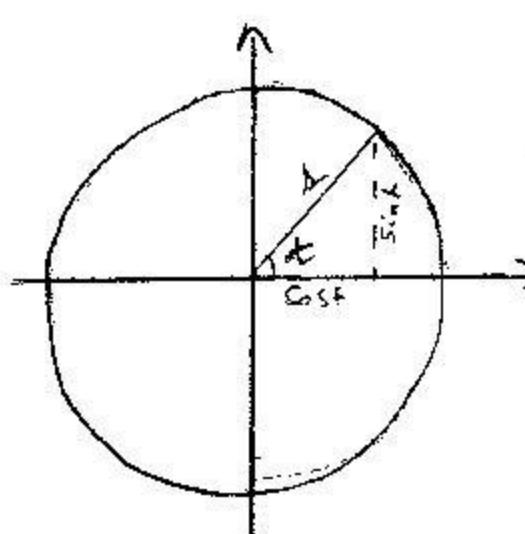


$\mathbb{R}^n \rightarrow \text{M/N}$ 

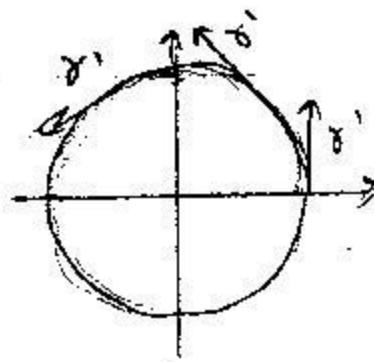
Defn kira mba $f: [a, b] \rightarrow \mathbb{R}^n$ 3.3 p. 12
 sink π $0 \leq t \leq 2\pi$ $f(t) = (\cos t, \sin t)$ - ANS
 \Leftarrow 1 dim. fcn

$\gamma(t) = (x_1(t), x_2(t), \dots, x_n(t))$ - per. t $\gamma(t)$ defn \Rightarrow

$x_i: \mathbb{R} \rightarrow \mathbb{R}$ defn

לפ. x_i \Rightarrow x_i \in rk $a \leq t \leq b$ 3.3 p. 12 defn δ
defn δ \in \mathbb{R}^{3n+1} $\in \mathbb{R}^{3n+1}$ $\in \mathbb{R}^{3n+1}$ $\delta(t) = (x_1(t), x_2(t), \dots, x_n(t))$

$\delta'(t) = (-\sin t, \cos t) \Leftarrow 0 \leq t \leq 2\pi \quad \gamma(t) = (\cos t, \sin t)$



$\delta \perp \delta'$ defn

$\text{defn } \delta \perp \delta' \Leftrightarrow |\delta'(t)|$

$(\delta \perp \delta')$

$\frac{d}{dt} \langle \delta(t), \eta(t) \rangle = \langle \delta(t), \eta'(t) \rangle + \langle \delta'(t), \eta(t) \rangle$ defn η \in \mathbb{R}^n \Rightarrow η \perp δ defn

$$\frac{d}{dt} \langle \delta(t), \eta(t) \rangle = \frac{d}{dt} \sum_{i=1}^n \delta_i(t) \eta_i(t) = \sum_{i=1}^n \frac{d}{dt} \{ \delta_i(t) \eta_i(t) \} \stackrel{\text{defn}}{=} \text{defn}$$

$$\stackrel{\text{defn}}{=} \sum_{i=1}^n (\delta'_i(t) \eta_i(t) + \eta'_i(t) \delta_i(t)) = \sum_{i=1}^n \delta'_i(t) \eta_i(t) + \sum_{i=1}^n \eta'_i(t) \delta_i(t) = \langle \delta', \eta \rangle + \langle \delta, \eta' \rangle$$

- $\delta \perp \delta'$ - \Rightarrow $\text{defn } \delta \perp \delta' \Rightarrow \langle \delta, \eta \rangle = 0$

$$\frac{d}{dt} |\delta(t)|^2 = \frac{d}{dt} \langle \delta(t), \delta(t) \rangle = 2 \langle \delta(t), \delta'(t) \rangle = 0$$

$\text{defn } |\delta(t)|$ defn

$\text{defn } \{ |x| = a \}$ $\in \mathbb{R}^n$ δ

$\text{defn } \delta \perp \delta' \Leftrightarrow |\delta'(t)| \neq 0$

$\text{defn } \delta \perp \delta' \Leftrightarrow |\delta'(t)| \neq 0$

motion along

motion along $x, y \in A$ for the motion path $\gamma: I \rightarrow A$ is

$$\cdot \gamma(b) = y, \quad \gamma(a) = x \quad - \text{means } y = x \text{ and motion path } \gamma: [a, b] \rightarrow A$$

$$\begin{aligned} \text{for every point } p \in A, \quad & \gamma(p) = p \Leftrightarrow p \in [a, b] \\ \text{and every point } x, y \in A, \quad & \gamma(x) = y \Leftrightarrow x = y \text{ and } \gamma([x, y]) = [y, x] \\ \text{and every point } z \in A, \quad & \gamma(z) = z \end{aligned}$$

$$[x, y] = \{tx + (1-t)y \mid 0 \leq t \leq 1\} \quad \text{for } x, y \in S^{n-1} \quad \text{for } n \geq 2$$

$$\gamma(t) = \frac{tx + (1-t)y}{\|tx + (1-t)y\|} \quad \text{for } x, y \in S^{n-1}, \quad \text{for } n \geq 2$$

$$\cdot \gamma(1) = y, \quad \gamma(0) = x \quad - \text{means } (0 \notin [x, y]) \quad \text{for } n \geq 2$$

$$\cdot \text{every motion path } \gamma \text{ for } 0 \leq t \leq 1 \text{ has } |\gamma(t)| = 1$$

$$\left\{ \begin{array}{l} z \neq x \\ z \neq y \end{array} \right. : z \text{ not on segment } : x = -y \Leftrightarrow 0 \in [x, y] \quad \text{for } n \geq 2$$

$$- \text{motion along } S^{n-1} \quad \left\{ \begin{array}{l} z \neq \pm x \\ z \neq \pm y \end{array} \right. \quad \text{for } n \geq 2$$

$$z \in S^{n-1} \quad \alpha: [0, 1] \rightarrow S^{n-1}$$

$$y \in S^{n-1} \quad \beta: [0, 1] \rightarrow S^{n-1}$$

$$\cdot \beta(1) = y, \quad \beta(0) = z, \quad \alpha(1) = z, \quad \alpha(0) = x \quad - \text{e.g. } \gamma$$

$$\gamma(t) = \begin{cases} \alpha(t) & 0 \leq t \leq 1 \\ \beta(t-1) & 1 \leq t \leq 2 \end{cases} \quad \text{for } S^{n-1} \quad \text{for } n \geq 2$$

$$y \in S^{n-1} \quad \text{for motion } \gamma$$

position function

$\gamma: I \rightarrow A$ motion path $A \in \mathbb{R}^n$ for $f: A \rightarrow \mathbb{R}$ is

$$\cdot f(\gamma(t)) = 0 \quad - \text{e.g. } \gamma \in A \text{ and } f(\gamma) < 0, \quad f(\gamma) > 0 \quad - \text{e.g. } \gamma \in A$$

$$\cdot \gamma: [a, b] \rightarrow A \quad \text{motion path } \gamma \text{ for position function } f: A \rightarrow \mathbb{R}$$

$$\cdot \varphi(t) = f(\gamma(t)) \quad \text{for } f(y) < 0, \quad \varphi(b) = f(y) < 0, \quad \varphi(a) = f(x) > 0$$

so the path γ is

increasing, decreasing

and f is

continuous at y

and f is

continuous at x

$$\underset{z \in A}{\text{F}}(z(t_0)) = \underset{z \in A}{\text{F}}(t_0) = 0 \quad \text{and} \quad a < t_0 < b \quad \text{implies} \quad \text{F}(z) = 0 \quad \forall z \in N \quad z = z(t_0) \quad \text{and}$$

(continuity of F) implies $\text{F}(z)$ is continuous

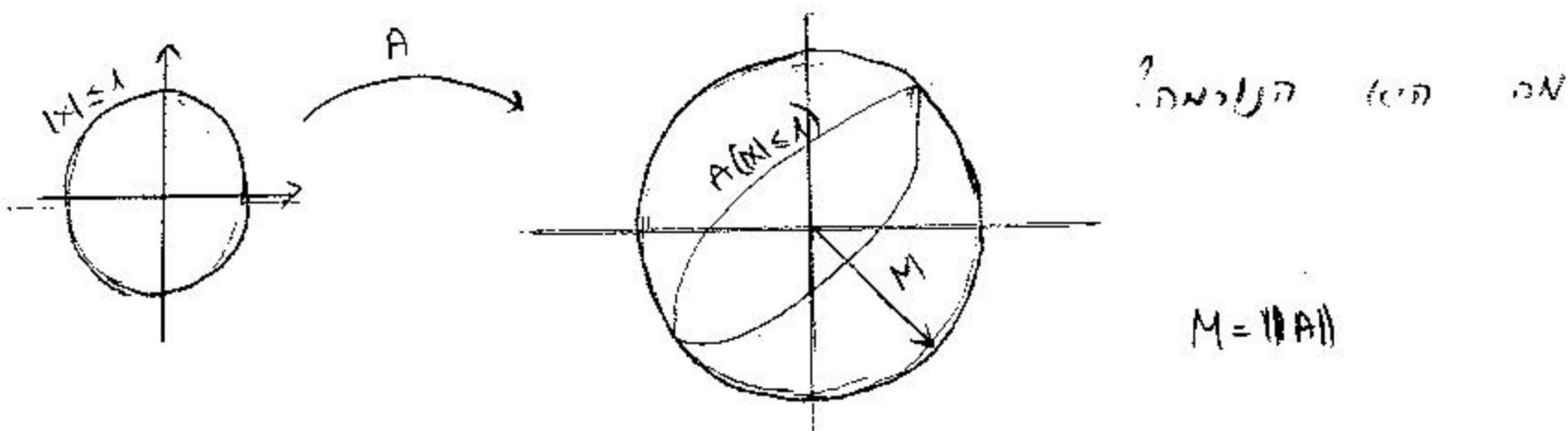
$$A: \mathbb{R}^n \rightarrow \mathbb{R}^{m \times n} \quad \|A\|: \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \text{def} \quad \text{such that} \quad (\forall x \in \mathbb{R}^n) \quad \|Ax\| \geq 0 \quad \text{and} \quad \|Ax\| = \max_{\|x\|=1} \|Ax\|$$

$$(m \times n \quad \text{matrix}) \quad \text{norm} \quad \text{def} \quad \text{such that} \quad \|Ax\| = \max_{\|x\|=1} |Ax|$$

$$\|A\| = \max_{\|x\|=1} |Ax| \quad \text{def} \quad \text{such that} \quad \|Ax\| = \max_{\|x\|=1} |Ax|$$

($\exists \lambda \in \mathbb{R}$ such that $\|Ax\| \leq \lambda \|x\| \quad \forall x \in \mathbb{R}^n$)

$\Rightarrow \exists M > 0$ such that $\|Ax\| \leq M \|x\| \quad \forall x \in \mathbb{R}^n$



$$|Ax| \leq \|A\| \leq \|A\| \cdot |x| \quad : |x|=1$$

$$|Ax| \leq \|A\| \cdot |x| \quad : x \in \mathbb{R}^n \quad \text{if} \quad \exists M > 0 \quad \text{such that} \quad |Ax| \leq M |x| \quad \forall x \in \mathbb{R}^n$$

continuity of $\|A\|$, $|Ax_0| = \|A\| \quad \rightarrow \exists \delta > 0 \quad \text{such that} \quad |x - x_0| < \delta \Rightarrow |Ax - Ax_0| < \epsilon$

$$\|A\| = \max \{ \sqrt{\lambda}, \|A^t A\| \quad \text{where} \quad \lambda \geq 0 \}$$

($\lambda \geq 0$ such that $\lambda \geq \lambda_1 \geq \dots \geq \lambda_n$ and $\lambda_1 > 0$)

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ (calculated by eigenvalues of $A^t A$)

orthonormal basis $\{v_1, \dots, v_n\}$ such that

$$A v_i = \lambda_i v_i \quad ; \quad \langle v_i, v_j \rangle = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

$$\langle A^t A x, x \rangle = \langle A x, A x \rangle = \|Ax\|^2 \geq 0$$

$\lambda_n \geq \lambda_{n-1} \geq \dots \geq \lambda_1 \geq 0$ (calculated by eigenvalues of $A^t A$)

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle A^t A x, x \rangle$$

$A^t = A^*$

$v_1, \dots, v_n \rightarrow A^t A$ se für $0 \leq \lambda_1 \leq \dots \leq \lambda_n$ ist es möglich dann

durch λ_i se einzig eindeutig eindeutig ist

$$\text{d.h. } x = \sum_{i=1}^n \alpha_i v_i \text{ für } \alpha_1, \dots, \alpha_n \in \mathbb{R} \text{ eindeutig}$$

$$|x|^2 = \left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{j=1}^n \alpha_j v_j \right\rangle = \sum_{i=1}^n \alpha_i^2 \langle v_i, v_i \rangle + \sum_{i \neq j} \alpha_i \alpha_j \langle v_i, v_j \rangle = \sum_{i=1}^n \alpha_i^2$$

$$|Ax|^2 = \left\langle A^t A \left(\sum_{i=1}^n \alpha_i v_i \right), \sum_{j=1}^n \alpha_j v_j \right\rangle = \left\langle \sum_{i=1}^n \alpha_i \underbrace{A^t A v_i}_{\lambda_i v_i}, \sum_{j=1}^n \alpha_j v_j \right\rangle = \sum_{i=1}^n \lambda_i \alpha_i^2 \leq$$

$$\begin{aligned} \Leftrightarrow \max |\lambda_i| \cdot \left(\sum_{i=1}^n \alpha_i^2 \right) &\leq \max |\lambda_i| = \lambda_n \\ &= |x|^2 = 1 \end{aligned} \quad - \text{ falls } x = v_n \text{ f.s.}$$

$$|Av_n|^2 = \left\langle A^t A v_n, v_n \right\rangle = \lambda_n \langle v_n, v_n \rangle = \lambda_n$$

$$\|A\|^2 = \max_{\|x\|=1} |Ax|^2 \leq \lambda_n \quad \text{p/n}$$

$$\|A\| = \sqrt{\lambda_n} = \max \left\{ \sqrt{\lambda_i} : A^t A \text{ se } \rightarrow \lambda_i \right\} \quad - \text{ s.o.}$$

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{3}x + y \\ 2y \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{3} & 1 \\ 0 & 2 \end{pmatrix}}_{A^t} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{det} = 2\sqrt{3} = 2\sqrt{3}/2$$

$$\therefore \text{RNB! } \lambda_1 = 2 \quad \lambda_2 = \sqrt{3} \quad A^t A = \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 5 \end{pmatrix}$$

$$8 = \text{tr}(A^t A) = \lambda_1 + \lambda_2$$

$$: 2 \rightarrow 30N \text{ mit } A^t A$$

$$12 = |\det(A^t A)| = \lambda_1 \cdot \lambda_2$$

$$\therefore \|A\| = \sqrt{6} \quad \Leftarrow \quad \lambda_2 = 2, \lambda_1 = 6 \quad \Leftarrow$$

- 2, 3, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, 120, 126, 132, 138, 144, 150, 156, 162, 168, 174, 180, 186, 192, 198, 204, 210, 216, 222, 228, 234, 240, 246, 252, 258, 264, 270, 276, 282, 288, 294, 200, 205, 210, 215, 220, 225, 230, 235, 240, 245, 250, 255, 260, 265, 270, 275, 280, 285, 290, 295, 300, 305, 310, 315, 320, 325, 330, 335, 340, 345, 350, 355, 360, 365, 370, 375, 380, 385, 390, 395, 400, 405, 410, 415, 420, 425, 430, 435, 440, 445, 450, 455, 460, 465, 470, 475, 480, 485, 490, 495, 500, 505, 510, 515, 520, 525, 530, 535, 540, 545, 550, 555, 560, 565, 570, 575, 580, 585, 590, 595, 600, 605, 610, 615, 620, 625, 630, 635, 640, 645, 650, 655, 660, 665, 670, 675, 680, 685, 690, 695, 700, 705, 710, 715, 720, 725, 730, 735, 740, 745, 750, 755, 760, 765, 770, 775, 780, 785, 790, 795, 800, 805, 810, 815, 820, 825, 830, 835, 840, 845, 850, 855, 860, 865, 870, 875, 880, 885, 890, 895, 900, 905, 910, 915, 920, 925, 930, 935, 940, 945, 950, 955, 960, 965, 970, 975, 980, 985, 990, 995, 1000, 1005, 1010, 1015, 1020, 1025, 1030, 1035, 1040, 1045, 1050, 1055, 1060, 1065, 1070, 1075, 1080, 1085, 1090, 1095, 1100, 1105, 1110, 1115, 1120, 1125, 1130, 1135, 1140, 1145, 1150, 1155, 1160, 1165, 1170, 1175, 1180, 1185, 1190, 1195, 1200, 1205, 1210, 1215, 1220, 1225, 1230, 1235, 1240, 1245, 1250, 1255, 1260, 1265, 1270, 1275, 1280, 1285, 1290, 1295, 1300, 1305, 1310, 1315, 1320, 1325, 1330, 1335, 1340, 1345, 1350, 1355, 1360, 1365, 1370, 1375, 1380, 1385, 1390, 1395, 1400, 1405, 1410, 1415, 1420, 1425, 1430, 1435, 1440, 1445, 1450, 1455, 1460, 1465, 1470, 1475, 1480, 1485, 1490, 1495, 1500, 1505, 1510, 1515, 1520, 1525, 1530, 1535, 1540, 1545, 1550, 1555, 1560, 1565, 1570, 1575, 1580, 1585, 1590, 1595, 1600, 1605, 1610, 1615, 1620, 1625, 1630, 1635, 1640, 1645, 1650, 1655, 1660, 1665, 1670, 1675, 1680, 1685, 1690, 1695, 1700, 1705, 1710, 1715, 1720, 1725, 1730, 1735, 1740, 1745, 1750, 1755, 1760, 1765, 1770, 1775, 1780, 1785, 1790, 1795, 1800, 1805, 1810, 1815, 1820, 1825, 1830, 1835, 1840, 1845, 1850, 1855, 1860, 1865, 1870, 1875, 1880, 1885, 1890, 1895, 1900, 1905, 1910, 1915, 1920, 1925, 1930, 1935, 1940, 1945, 1950, 1955, 1960, 1965, 1970, 1975, 1980, 1985, 1990, 1995, 2000, 2005, 2010, 2015, 2020, 2025, 2030, 2035, 2040, 2045, 2050, 2055, 2060, 2065, 2070, 2075, 2080, 2085, 2090, 2095, 2100, 2105, 2110, 2115, 2120, 2125, 2130, 2135, 2140, 2145, 2150, 2155, 2160, 2165, 2170, 2175, 2180, 2185, 2190, 2195, 2200, 2205, 2210, 2215, 2220, 2225, 2230, 2235, 2240, 2245, 2250, 2255, 2260, 2265, 2270, 2275, 2280, 2285, 2290, 2295, 2300, 2305, 2310, 2315, 2320, 2325, 2330, 2335, 2340, 2345, 2350, 2355, 2360, 2365, 2370, 2375, 2380, 2385, 2390, 2395, 2400, 2405, 2410, 2415, 2420, 2425, 2430, 2435, 2440, 2445, 2450, 2455, 2460, 2465, 2470, 2475, 2480, 2485, 2490, 2495, 2500, 2505, 2510, 2515, 2520, 2525, 2530, 2535, 2540, 2545, 2550, 2555, 2560, 2565, 2570, 2575, 2580, 2585, 2590, 2595, 2600, 2605, 2610, 2615, 2620, 2625, 2630, 2635, 2640, 2645, 2650, 2655, 2660, 2665, 2670, 2675, 2680, 2685, 2690, 2695, 2700, 2705, 2710, 2715, 2720, 2725, 2730, 2735, 2740, 2745, 2750, 2755, 2760, 2765, 2770, 2775, 2780, 2785, 2790, 2795, 2800, 2805, 2810, 2815, 2820, 2825, 2830, 2835, 2840, 2845, 2850, 2855, 2860, 2865, 2870, 2875, 2880, 2885, 2890, 2895, 2900, 2905, 2910, 2915, 2920, 2925, 2930, 2935, 2940, 2945, 2950, 2955, 2960, 2965, 2970, 2975, 2980, 2985, 2990, 2995, 3000, 3005, 3010, 3015, 3020, 3025, 3030, 3035, 3040, 3045, 3050, 3055, 3060, 3065, 3070, 3075, 3080, 3085, 3090, 3095, 3100, 3105, 3110, 3115, 3120, 3125, 3130, 3135, 3140, 3145, 3150, 3155, 3160, 3165, 3170, 3175, 3180, 3185, 3190, 3195, 3200, 3205, 3210, 3215, 3220, 3225, 3230, 3235, 3240, 3245, 3250, 3255, 3260, 3265, 3270, 3275, 3280, 3285, 3290, 3295, 3300, 3305, 3310, 3315, 3320, 3325, 3330, 3335, 3340, 3345, 3350, 3355, 3360, 3365, 3370, 3375, 3380, 3385, 3390, 3395, 3400, 3405, 3410, 3415, 3420, 3425, 3430, 3435, 3440, 3445, 3450, 3455, 3460, 3465, 3470, 3475, 3480, 3485, 3490, 3495, 3500, 3505, 3510, 3515, 3520, 3525, 3530, 3535, 3540, 3545, 3550, 3555, 3560, 3565, 3570, 3575, 3580, 3585, 3590, 3595, 3600, 3605, 3610, 3615, 3620, 3625, 3630, 3635, 3640, 3645, 3650, 3655, 3660, 3665, 3670, 3675, 3680, 3685, 3690, 3695, 3700, 3705, 3710, 3715, 3720, 3725, 3730, 3735, 3740, 3745, 3750, 3755, 3760, 3765, 3770, 3775, 3780, 3785, 3790, 3795, 3800, 3805, 3810, 3815, 3820, 3825, 3830, 3835, 3840, 3845, 3850, 3855, 3860, 3865, 3870, 3875, 3880, 3885, 3890, 3895, 3900, 3905, 3910, 3915, 3920, 3925, 3930, 3935, 3940, 3945, 3950, 3955, 3960, 3965, 3970, 3975, 3980, 3985, 3990, 3995, 4000, 4005, 4010, 4015, 4020, 4025, 4030, 4035, 4040, 4045, 4050, 4055, 4060, 4065, 4070, 4075, 4080, 4085, 4090, 4095, 4100, 4105, 4110, 4115, 4120, 4125, 4130, 4135, 4140, 4145, 4150, 4155, 4160, 4165, 4170, 4175, 4180, 4185, 4190, 4195, 4200, 4205, 4210, 4215, 4220, 4225, 4230, 4235, 4240, 4245, 4250, 4255, 4260, 4265, 4270, 4275, 4280, 4285, 4290, 4295, 4300, 4305, 4310, 4315, 4320, 4325, 4330, 4335, 4340, 4345, 4350, 4355, 4360, 4365, 4370, 4375, 4380, 4385, 4390, 4395, 4400, 4405, 4410, 4415, 4420, 4425, 4430, 4435, 4440, 4445, 4450, 4455, 4460, 4465, 4470, 4475, 4480, 4485, 4490, 4495, 4500, 4505, 4510, 4515, 4520, 4525, 4530, 4535, 4540, 4545, 4550, 4555, 4560, 4565, 4570, 4575, 4580, 4585, 4590, 4595, 4600, 4605, 4610, 4615, 4620, 4625, 4630, 4635, 4640, 4645, 4650, 4655, 4660, 4665, 4670, 4675, 4680, 4685, 4690, 4695, 4700, 4705, 4710, 4715, 4720, 4725, 4730, 4735, 4740, 4745, 4750, 4755, 4760, 4765, 4770, 4775, 4780, 4785, 4790, 4795, 4800, 4805, 4810, 4815, 4820, 4825, 4830, 4835, 4840, 4845, 4850, 4855, 4860, 4865, 4870, 4875, 4880, 4885, 4890, 4895, 4900, 4905, 4910, 4915, 4920, 4925, 4930, 4935, 4940, 4945, 4950, 4955, 4960, 4965, 4970, 4975, 4980, 4985, 4990, 4995, 5000, 5005, 5010, 5015, 5020, 5025, 5030, 5035, 5040, 5045, 5050, 5055, 5060, 5065, 5070, 5075, 5080, 5085, 5090, 5095, 5100, 5105, 5110, 5115, 5120, 5125, 5130, 5135, 5140, 5145, 5150, 5155, 5160, 5165, 5170, 5175, 5180, 5185, 5190, 5195, 5200, 5205, 5210, 5215, 5220, 5225, 5230, 5235, 5240, 5245, 5250, 5255, 5260, 5265, 5270, 5275, 5280, 5285, 5290, 5295, 5300, 5305, 5310, 5315, 5320, 5325, 5330, 5335, 5340, 5345, 5350, 5355, 5360, 5365, 5370, 5375, 5380, 5385, 5390, 5395, 5400, 5405, 5410, 5415, 5420, 54

1.1.1.7(1.3), 2.1.3

$$\text{Definition 2.1.3: } \forall a \in D \quad \exists L(a) \in \mathbb{R}^m \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall h \in \mathbb{R}^n \quad \text{such that } \|h\| < \delta \Rightarrow \|L(a)h - f(a+h)\| < \epsilon$$

$$\lim_{h \rightarrow 0} \frac{\|f(a+h) - f(a) - L(a)h\|}{\|h\|} = 0 \quad (\text{i})$$

(a+h \in D)

$$f(a+h) = f(a) + L(a)h + o(\|h\|) \quad (\text{for small } \|h\|)$$

From this we can see that the function f is continuous at a .

$$D_f(a) = D_f(a) \quad (\text{no such } L)$$

$$D_f(a)(h) = \lim_{t \rightarrow 0} \frac{f(a+th) - f(a)}{t} \quad (h \in \mathbb{R}^n) \quad (\text{ii})$$

$$F = (f_1, \dots, f_m) : a \mapsto \begin{cases} f_i : a \mapsto f_i(a) \end{cases} \quad F : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \text{etc}$$

$$\frac{\partial F_i}{\partial x_j} : \begin{matrix} \text{continuous} \\ 1 \leq i \leq m, 1 \leq j \leq n \end{matrix} \quad a \mapsto \begin{cases} \frac{\partial f_i}{\partial x_j}(a) \end{cases} \quad F : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$D_F(a) = \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \quad (\text{iii})$$

Example 1.2.1: If f is differentiable at a , then $D_f(a)$ is called the derivative of f at a .

$$f(x,y) = \begin{cases} \frac{x^3+2y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$|f(x,y) - f(0,0)| = \left| \frac{x^3+2y^3}{x^2+y^2} \right| \leq \frac{|x|^3 + |2y|^3}{x^2+y^2} \leq \frac{|x|^3}{x^2} + \frac{2|y|^3}{y^2} = |x| + 2|y| \leq C((x,y)) \rightarrow (0,0) \quad (\text{if } (x,y) \neq (0,0))$$

1.2.3: Uniqueness

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \quad \text{etc} \quad (\text{iv})$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \rightarrow 0} \frac{x}{x} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = 2$$

$$D_f(0,0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)(0,0) = (1,2) \quad (\text{v})$$

$$\frac{|f(x,y) - f(0,0) - (1,2) \cdot \begin{pmatrix} x \\ y \end{pmatrix}|}{\sqrt{x^2+y^2}} \xrightarrow{\text{?}} 0 \quad (\text{vi})$$

$$\Rightarrow \frac{\left| \frac{x^3+2y^3}{x^2+y^2} - 0 - x - 2y \right|}{\sqrt{x^2+y^2}} \xrightarrow{\text{?}} 0 \quad (\text{vii})$$

$$\frac{\left| \frac{3x^3}{2x^2} - 3x \right|}{\sqrt{2}|x|} = \frac{3|x|}{\sqrt{2}|x|} = \frac{3}{2\sqrt{2}} \rightarrow 0$$

- If $x \neq 0$ then $\lim_{x \rightarrow 0}$ is 0
 (as $|y| > 0$ implies $y \neq 0$ and $y \neq 0$)

∴ $f(x,y) \rightarrow 0$ as $(x,y) \rightarrow (0,0)$

$$f(x,y) = \begin{cases} (x^2+y^2) \sin \frac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$

$$\frac{|f(x,y) - f(0,0) - (0,0) \begin{pmatrix} x \\ y \end{pmatrix}|}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2} \sin \left(\frac{1}{\sqrt{x^2+y^2}} \right)}{\sqrt{x^2+y^2}} \xrightarrow{(x,y) \rightarrow (0,0)} 0$$

∴ $f(x,y) \rightarrow 0$ as $(x,y) \rightarrow (0,0)$